5 LINEAR PROGRAMMING

Objectives

After studying this chapter you should

- be able to formulate linear programming problems from contextual problems;
- be able to identify feasible regions for linear programming problems;
- be able to find solutions to linear programming problems using graphical means;
- be able to apply the simplex method using slack variables;
- understand the simplex tableau procedure.

5.0 Introduction

The methods of linear programming were originally developed between 1945 and 1955 by American mathematicians to solve problems arising in industry and economic planning. Many such problems involve constraints on the size of the workforce, the quantities of raw materials available, the number of machines available and so on. The problems that will be solved usually have two variables in them and can be solved graphically, but problems occurring in industry have many more variables and have to be solved by computer. For example, in oil refineries, problems arise with hundreds of variables and tens of thousands of constraints.

Another application is in determining the best diet for farm animals such as pigs. In order to maximise the profit a pig farmer needs to ensure that the pigs are fed appropriate food and sufficient quantities of it to produce lean meat. The pigs require a daily allocation of carbohydrate, protein, amino acids, minerals and vitamins. Each involves various components. For example, the mineral content includes calcium, phosphorus, salt, potassium, iron, magnesium, zinc, copper, manganese, iodine, and selenium. All these dietary constituents should be present, in correct amounts.

A statistician at Exeter University has devised a computer program for use by farmers and companies producing animal feeds which enables them to provide the right diet for pigs at various stages of development, such as the weaning, growing and finishing stages. The program involves 20 variables and 10 equations!

Undoubtably linear programming is one of the most widespread methods used to solve management and economic problems, and has been applied in a wide variety of situations and contexts.

5.1 Formation of linear programming problems

You are now in a position to use your knowledge of inequalities from the previous chapter to illustrate **linear programming** with the following case study.

Suppose a manufacturer of printed circuits has a stock of

200 resistors, 120 transistors and 150 capacitors

and is required to produce two types of circuits.

Type A requires 20 resistors, 10 transistors and 10 capacitors.

Type B requires 10 resistors, 20 transistors and 30 capacitors.

If the profit on type A circuits is £5 and that on type B circuits is £12, how many of each circuit should be produced in order to maximise the profit?

You will not actually solve this problem yet, but show how it can be formulated as a linear programming problem. There are three vital stages in the formulation, namely

(a) What are the unknowns?

(b) What are the constraints?

(c) What is the profit/cost to be maximised/minimised?

For this problem,

(a) **What are the unknowns?**

Clearly the number of type A and type B circuits produced; so we define

$x =$ number of type A circuits produced

$y =$ number of type B circuits produced

(b) **What are the constraints?**

There are constraints associated with the total number of resistors, transistors and capacitors available.

Resistors Since each type A requires 20 resistors,
and each type B requires 10 resistors,
then
$20x + 10y \le 200$,
as there is a total of 200 resistors available.

Transistors Similarly

$$10x + 20y \le 120$$

Capacitors Similarly

$$10x + 30y \le 150$$

Finally you must state the obvious (but nevertheless important) inequalities

$$x \ge 0,\ y \ge 0\,.$$

(c) **What is the profit?**

Since each type A gives £5 profit and each type B gives £12 profit, the total profit is £P, where

$$P = 5x + 12y\,.$$

You can now summarise the problem as:

maximise $P = 5x + 12y$

subject to
$$\begin{aligned} 20x + 10y &\le 200 \\ 10x + 20y &\le 120 \\ 10x + 30y &\le 150 \\ x &\ge 0 \\ y &\ge 0\,. \end{aligned}$$

This is called a **linear** programming problem since both the objective function P and the constraints are all linear in x and y.

In this particular example you should be aware that x and y can only be integers since it is not sensible to consider fractions of a printed circuit. In all linear programming problems you need to consider if the variables are integers.

Activity 1 Feasible solutions

Show that $x = 5, y = 3$ satisfies all the constraints.

Find the associated profit for this solution, and compare this profit with other possible solutions.

At this stage, you will not continue with finding the actual solutions but will concentrate on further practice in formulating problems of this type.

The key stage is the first one, namely that of identifying the unknowns; so you must carefully read the problem through in order to identify the basic unknowns. Once you have done this successfully, it should be straight forward to express both the constraints and the profit function in terms of the unknowns.

A further condition to note in many of the problems is that the unknowns must be positive **integers**.

Example

A small firm builds two types of garden shed.

Type A requires 2 hours of machine time and 5 hours of craftsman time.

Type B requires 3 hours of machine time and 5 hours of craftsman time.

Each day there are 30 hours of machine time available and 60 hours of craftsman time. The profit on each type A shed is £60 and on each type B shed is £84.

Formulate the appropriate linear programming problem.

Solution

(a) **Unknowns**

Define

$x =$ number of Type A sheds produced each day,

$y =$ number of Type B sheds produced each day.

(b) **Constraints**

Machine time: $2x + 3y \le 30$

Craftsman time: $5x + 5y \le 60$

and $x \ge 0, y \ge 0$

(c) **Profit**

$P = 60x + 84y$

So, in summary, the linear programming problem is

maximise $P = 60x + 84y$

$$\text{subject to } \begin{aligned} 2x + 3y &\le 30 \\ x + y &\le 12 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

Exercise 5A

1. Ann and Margaret run a small business in which they work together making blouses and skirts.

   Each blouse takes 1 hour of Ann's time together with 1 hour of Margaret's time. Each skirt involves Ann for 1 hour and Margaret for half an hour. Ann has 7 hours available each day and Margaret has 5 hours each day.

   They could just make blouses or they could just make skirts or they could make some of each.

   Their first thought was to make the same number of each. But they get £8 profit on a blouse and only £6 on a skirt.

   (a) Formulate the problem as a linear programming problem.

   (b) Find three solutions which satisfy the constraints.

2. A distribution firm has to transport 1200 packages using large vans which can take 200 packages each and small vans which can take 80 packages each. The cost of running each large van is £40 and of each small van is £20. Not more than £300 is to be spent on the job. The number of large vans must not exceed the number of small vans.

   Formulate this problem as a linear programming problem given that the objective is to **minimise** costs.

3. A firm manufactures wood screws and metal screws. All the screws have to pass through a threading machine and a slotting machine. A box of wood screws requires 3 minutes on the slotting machine and 2 minutes on the threading machine. A box of metal screws requires 2 minutes on the slotting machine and 8 minutes on the threading machine. In a week, each machine is available for 60 hours.

   There is a profit of £10 per box on wood screws and £17 per box on metal screws.

   Formulate this problem as a linear programming problem given that the objective is to **maximise** profit.

4. A factory employs unskilled workers earning £135 per week and skilled workers earning £270 per week. It is required to keep the weekly wage bill below £24 300.

   The machines require a minimum of 110 operators, of whom at least 40 must be skilled. Union regulations require that the number of skilled workers should be at least half the number of unskilled workers.

   If x is the number of unskilled workers and y the number of skilled workers, write down all the constraints to be satisfied by x and y.

5.2 Graphical solution

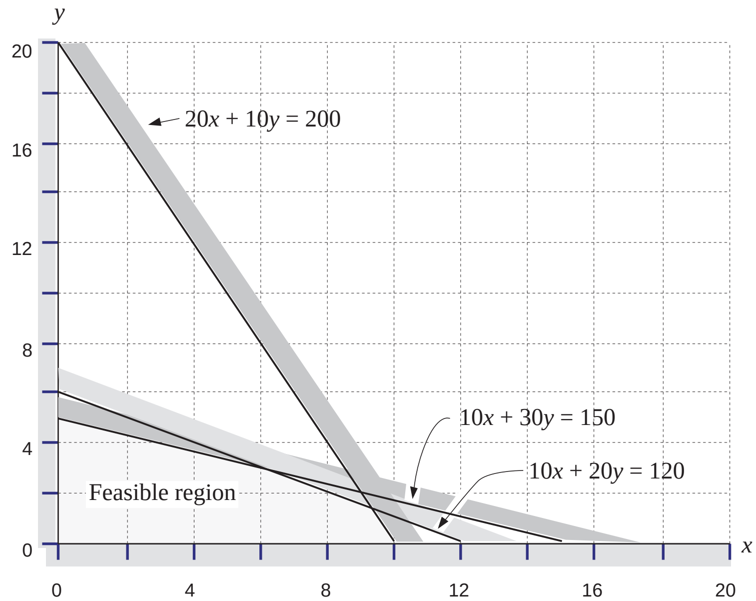
In the previous section you worked through problems that led to a linear programming problem in which a **linear** function of x and y is to be maximised (or minimised) subject to a number of **linear** inequalities to be satisfied.

Fortunately problems of this type with just two variables can easily be solved using a graphical method. The method will first be illustrated using the example from the text in Section 5.1. This resulted in the linear programming problem

$$\text{maximise } P = 5x + 12y$$

$$\begin{aligned} \text{subject to } \quad 20x + 10y &\le 200 \\ 10x + 20y &\le 120 \\ 10x + 30y &\le 150 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

You can illustrate the **feasible** (i.e. allowable) region by graphing all the inequalities and shading out the regions not allowed . This is illustrated in the figure below.



Magnifying the feasible region, you can look at the family of straight lines defined by

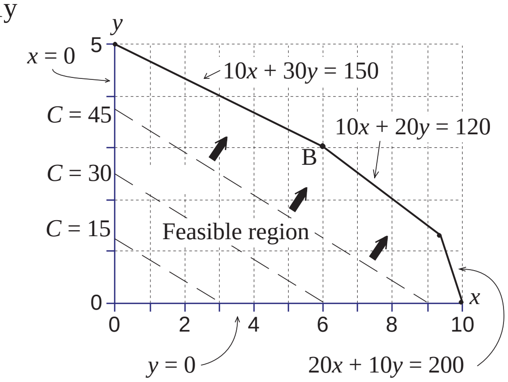
$$C = 5x + 12y$$

where C takes various values.

The figure shows, for example, the lines defined by

$$C = 15,\ C = 30,\ \text{and } C = 45.$$

On each of these lines any point gives the same profit.



Activity 2

Check that the points

$$x = 1,\ y = \tfrac{25}{12}$$
$$x = 2,\ y = \tfrac{5}{3}$$
$$x = 4,\ y = \tfrac{5}{6}$$

each lie on the line defined by $C = 30$. What profit does each of these points give?

Where is the point representing maximum profit ?

As the profit line moves to the right, the profit increases and so the maximum profit corresponds to the last point touched as the profit line moves out of the feasible region. This is the point B, the intersection of

$$10x + 30y = 150 \text{ and } 10x + 20y = 120$$

Solving these equations gives $10y = 30$, i.e. $y = 3$ and $x = 6$. So maximum profit occurs at the point (6, 3) and the profit is given by

$$P = 5 \times 6 + 12 \times 3 = 66.$$

Example

A farmer has 20 hectares for growing barley and swedes. The farmer has to decide how much of each to grow. The cost per hectare for barley is £30 and for swedes is £20. The farmer has budgeted £480.

Barley requires 1 man-day per hectare and swedes require 2 man-days per hectare. There are 36 man-days available.

The profit on barley is £100 per hectare and on swedes is £120 per hectare.

Find the number of hectares of each crop the farmer should sow to maximise profits.

Solution

The problem is formulated as a linear programming problem:

(a) **Unknowns**

$x =$ number of hectares of barley

$y =$ number of hectares of swedes

(b) **Constraints**

Land	$x + y \le 20$
Cost	$30x + 20y \le 480$
Manpower	$x + 2y \le 36$

(c) **Profit**

$$P = 100x + 120y$$

To summarise, maximise $P = 100x + 120y$

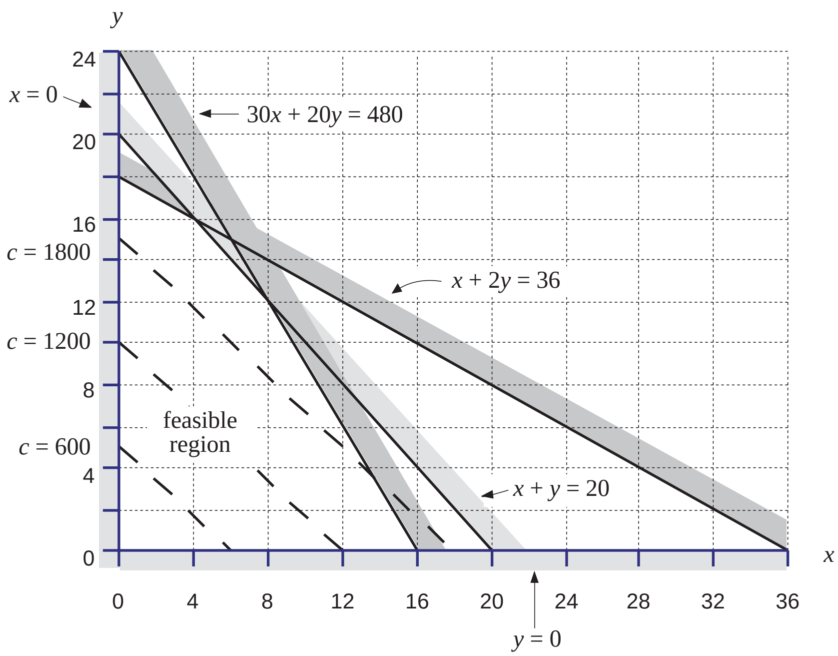
$$\begin{aligned} \text{subject to } x + y &\le 20 \\ 30x + 20y &\le 480 \\ x + 2y &\le 36 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

The feasible region is identified by the region enclosed by the five inequalities, as shown below. The profit lines are given by

$$C = 100x + 120y$$

and again you can see that C increases as the line (shown dotted) moves to the right. Continuing in this way, the maximum profit will occur at the intersection of

$$x + 2y = 36 \text{ and } x + y = 20$$



At this point $x = 4$ and $y = 16$, and the corresponding maximum profit is given by

$$P = 100 \times 4 + 120 \times 16 = 2320$$

The farmer should sow 4 hectares with barley and 16 with swedes.

Exercise 5B

1. Solve the linear programming problem defined in Question 1 of Exercise 5A.
2. Solve the linear programming problem defined in Question 2 of Exercise 5A.
3. A camp site for caravans and tents has an area of 1800m^2 and is subject to the following regulations:

   The number of caravans must not exceed 6.

   Reckoning on 4 persons per caravan and 3 per tent, the total number of persons must not exceed 48.

   At least 200 m^2 must be available for each caravan and 90 m^2 for each tent.

   The nightly charges are £2 for a caravan and £1 for a tent.

   Find the greatest possible nightly takings.

   How many caravans and tents should be admitted if the site owner wants to make the maximum profit and have

   (a) as many caravans as possible,

   (b) as many tents as possible?

4. The annual subscription for a tennis club is £20 for adults and £8 for juniors. The club needs to raise at least £800 in subscriptions to cover its expenses.

   The total number of members is restricted to 50. The number of junior members is to be between one quarter and one third of the number of adult members.

   Represent the information graphically and find the numbers of adult and junior members which will bring in the largest amount of money in subscriptions.

   Find also the least total membership which will satisfy the conditions.

5. The numbers of units of vitamins A, B and C in a kilogram of foods X and Y are as follows:

Food	Vitamin A	Vitamin B	Vitamin C
X	5	2	6
Y	4	6	2

   A mixture of the two foods is made which has to contain at least 20 units of vitamin A, at least 24 units of vitamin B and at least 12 units of vitamin C.

   Find the smallest total amount of X and Y to satisfy these constraints.

   Food Y is three times as expensive as Food X. Find the amounts of each to minimise the cost and satisfy the constraints.

5.3 Simplex method

Where will a linear programming solution always occur?

Looking back at the second example in Section 5.2, the slope of the profit line was $\frac{5}{6}$. (The slope is actually negative but it is sufficient to just consider the magnitude of the slopes of the lines.) This is more than the slope of the line $x+2y=36$ (namely $\frac{1}{2}$), but less than the slope of the other two lines, $x+y=20$ (i.e. 1) and $30x+20y=480$ (i.e. $\frac{3}{2}$).

So the solution will occur at the intersection of the two lines with slopes $\frac{1}{2}$ and 1.

Activity 3

Check the slopes of the constraints and profit function in the first example in the text in Section 5.2.

Another point worth noting here is that the solution of a linear programming problem will occur, in general, at one of the **vertices** of the feasible region if either non-integer solutions are acceptable or the vertex in question happens to have integral coordinates.

So an alternative to the graphical method of solution would be to

(a) find all the vertices of the feasible region;

(b) find the value of the profit function at each of these vertices;

(c) choose the one which gives maximum value to the profit function.

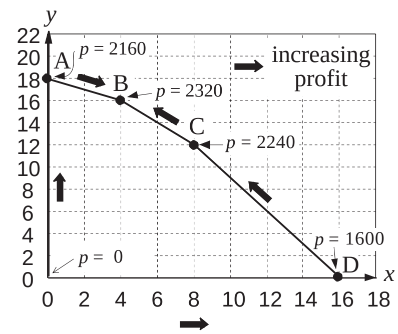
(We have not considered the possibility of non-integer vertices, which may not make sense in terms of the original problem.)

You can see how this method works with the second example in Section 5.2. The vertices are given by

(a) 0 (0, 0)
(b) A (0, 18)
(c) B (4, 16)
(d) C (8, 12)
(e) D (16, 0)

and the corresponding profits in £ are

Point	Profit
0	0
A	2160
B	2320
C	2240
D	1600



As you can see, as you move round the feasible region, the profit increases from 0 to A to B, but then decreases to C to D and back to 0.

In more complicated problems, it is helpful to introduce the idea of **slack variables**. For the problem above, with the three inequalities

$$x + y \le 20$$
$$30x + 20y \le 480$$
$$x + 2y \le 36$$

three new variables are defined by

$$r = 20 - x - y$$
$$s = 480 - 30x - 20y$$
$$t = 36 - x - 2y$$

The three inequalities can now be written as

$$r \ge 0,\ s \ge 0,\ t \ge 0$$

as well as $x \ge 0,\ y \ge 0$. The variables r, s and t are called the slack variables as they represent the amount of slack between the total quantity available and how much is being used.

The importance of the slack variables is that you can now define each vertex in terms of two of the variables, x, y, r, s, or t, being zero; for example, at A,

$$x = t = 0.$$

Activity 4

Complete the table below, defining each vertex

0	$x = y = 0$
A	$t = x = 0$
B	....
C	....
D	....

The procedure of increasing the profit from one vertex to the next will be again followed. Starting at the origin

$$P = 100x + 120y = 0 \text{ at } x = y = 0$$

Clearly P will increase in either direction from the origin - moving up the y axis means that x is held at zero whilst y increases. Although you know from the diagram that the next vertex reached will be A, how could you work that out without a picture? As x is being kept at zero and y is increasing the next vertex met will either be where $x = r = 0$ or where $x = s = 0$ or where $x = t = 0$. But note that

$$x = r = 0 \Rightarrow y = 20$$
$$x = s = 0 \Rightarrow y = 24$$
$$x = t = 0 \Rightarrow y = 18 \leftarrow \text{ smallest}$$

and so as y increases the first of these three points it reaches is the one where y is the smallest, namely $y = 18$ and the vertex is where $x = t = 0$ (which is the point A, thus confirming without a picture what has already been seen). You can now express the profit P in terms of x and t:

$$\begin{aligned} P &= 100x + 120y \\ &= 100x + 120\frac{(36 - x - t)}{2} \\ &= 40x - 60t + 2160 \end{aligned}$$

How can you tell if P will increase as x increases?

So you now increase x, keeping t at zero. You will next meet a vertex where either $t = s = 0$, $t = r = 0$ or $t = y = 0$. As x is increasing the vertex will be the one of those three points where x is the smallest:

$$t = s = 0 \Rightarrow x = 6$$
$$t = r = 0 \Rightarrow x = 4 \leftarrow \text{ smallest}$$
$$t = y = 0 \Rightarrow x = 36$$

So the next vertex reached is where $t = r = 0$, (which is B) and P is now expressed in terms of t and r. To do this eliminate y from the first and third of the original equations by noting that

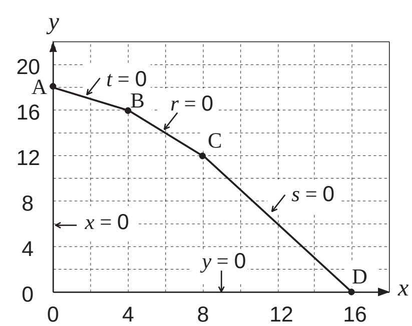
$$2r - t = 2(20 - x - y) - (36 - x - 2y) = 4 - x$$

Hence

$$\begin{aligned} P &= 40x - 60t + 2160 \\ &= 40(4 - 2r + t) - 60t + 2160 \\ &= -80r - 20t + 2320 \end{aligned}$$

How can you tell that P has reached its maximum?

Throughout the feasible region you know that $r \geq 0$ and $t \geq 0$ and so it is clear from the negative coefficients in the above expression for P that P reaches its maximum value of 2320 when r and t are both zero. This happens when $x = 4$ and $y = 16$.



Activity 5

Now travel round the feasible region from 0 to D to C to B. At each vertex, express P in terms of the defining variables, and check that P will continue to increase until B is reached.

Although this probably looks a much more complicated way of solving linear programming problems, its real application is to problems of more than 2 variables. These cannot be solved graphically, but can be solved using a procedure using slack variables called the **simplex** method.

Exercise 5C

1. (a) Solve the linear programming problem
   maximise $P = 2x + 4y$
   subject to
   $$\begin{aligned} x + 5y &\leq 10 \\ 4x + y &\leq 8 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$
   by a graphical method.

   (b) Introduce slack variables r and s, and solve the problem by the simplex method.

2. (a) Determine the vertices of the feasible region for the linear programming problem
   maximise $P = x + y$
   subject to
   $$\begin{aligned} x + 4y &\leq 8 \\ 2x + 3y &\leq 12 \\ 3x + y &\leq 9 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$
   Hence find the solution.

   (b) Verify this solution by using the simplex method.

3. Use the simplex method to solve the linear programming problem

maximise $P = 10x + 15y$

$$\begin{aligned} \text{subject to} \quad 4y + 10x &\le 40 \\ 10y + 3x &\le 30 \\ 5y + 4x &\le 20 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

*5.4 Simplex tableau

The way this method works will be illustrated with the example.

Maximise $P = x + 2y$

$$\begin{aligned} \text{subject to} \quad x + 4y &\le 20 \\ x + y &\le 8 \\ 5x + y &\le 32 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

As usual introduce slack variables r, s and t defined by

$$\begin{aligned} x + 4y + r &= 20 \\ x + y + s &= 8 \\ 5x + y + t &= 32 \end{aligned}$$

and write the equations in the matrix form

$$\begin{aligned} P - x - 2y \qquad\qquad &= 0 \\ x + 4y + r \qquad &= 20 \\ x + y \quad + s \quad &= 8 \\ 5x + y \qquad + t &= 32 \end{aligned}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & -1 & -2 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 5 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P \\ x \\ y \\ r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 20 \\ 8 \\ 32 \end{bmatrix}$$

The **augmented matrix** with the extra right hand column will be used.

P	***x***	***y***	***r***	***s***	***t***			**Comments**
1	−1	−2	0	0	0	0		Increase x first (y could have been chosen) and compare values of x where $y=r=0$, $y=s=0$, and $y=t=0$, namely $20/1, 8/1, 32/5$*. These values are easily spotted from the matrix as the R.H. figures divided by the corresponding coefficients of x. (*This term has the smallest positive value so now manipulate the matrix to express P in terms of y and t)
0	1	4	1	0	0	20		
0	1	1	0	1	0	8		
0	5	1	0	0	1	32		
1	−1	−2	0	0	0	0		
0	1	4	1	0	0	20		
0	1	1	0	1	0	8		
0	1	$\frac{1}{5}$	0	0	$\frac{1}{5}$	$\frac{32}{5}$	$\leftarrow R_4/5$	
1	0	$-\frac{9}{5}$	0	0	$\frac{1}{5}$	$\frac{32}{5}$	$\leftarrow R_1+R_4$	From the first row express P in terms of y and t, with a positive coefficient of y. Increase y and compare the values obtained from the figures in the R.H. column divided by the corresponding coefficients of y, namely $(68/5)/(19/5)$, $(8/5)/(4/5)$*, $(32/5)/(1/5)$. (*This term has the smallest positive value (where $s=t=0$) so now manipulate the matrix to express P in terms of s and t)
0	0	$\frac{19}{5}$	1	0	$-\frac{1}{5}$	$\frac{68}{5}$	$\leftarrow R_2-R_4$	
0	0	$\frac{4}{5}$	0	1	$-\frac{1}{5}$	$\frac{8}{5}$	$\leftarrow R_3-R_4$	
0	1	$\frac{1}{5}$	0	0	$\frac{1}{5}$	$\frac{32}{5}$		
1	0	$-\frac{9}{5}$	0	0	$\frac{1}{5}$	$\frac{32}{5}$		
0	0	$\frac{19}{5}$	1	0	$-\frac{1}{5}$	$\frac{68}{5}$		
0	0	1	0	$\frac{5}{4}$	$-\frac{1}{4}$	2	$\leftarrow R_3/\left(\frac{4}{5}\right)$	
0	1	$\frac{1}{5}$	0	0	$\frac{1}{5}$	$\frac{32}{5}$		
1	0	0	0	$\frac{9}{4}$	$-\frac{1}{4}$	10	$\leftarrow R_1+\frac{9}{5}R_3$	From the first row P could now be expressed in terms of s and t, with a positive coefficient of t. So now increase t and compare the values $6/(3/4)$*, $2/(-1/4)$, $6/(1/4)$ (*This term has the smallest positive value so now manipulate the matrix to express P in terms of r and s)
0	0	0	1	$-\frac{19}{4}$	$\frac{3}{4}$	6	$\leftarrow R_2-\frac{19}{5}R_3$	
0	0	1	0	$\frac{5}{4}$	$-\frac{1}{4}$	2		
0	1	0	0	$-\frac{1}{4}$	$\frac{1}{4}$	6	$\leftarrow R_4-\frac{1}{5}R_3$	
1	0	0	0	$\frac{9}{4}$	$-\frac{1}{4}$	10		
0	0	0	$\frac{4}{3}$	$-\frac{19}{3}$	1	8	$\leftarrow R_2/\left(\frac{3}{4}\right)$	
0	0	1	0	$\frac{5}{4}$	$-\frac{1}{4}$	2		
0	1	0	0	$-\frac{1}{4}$	$\frac{1}{4}$	6		
1	0	0	$\frac{1}{3}$	$\frac{2}{3}$	0	12	$\leftarrow R_1+\frac{1}{4}R_2$	From the first row P could now be expressed in terms of r and s, with a negative coefficient of each, so now stop; i.e. since the top row has all positive coefficients you can see that the maximum value of P is 12 and that it is reached when $r=s=0$ (which happens when $x=4$ and $y=4$).
0	0	0	$\frac{4}{3}$	$-\frac{19}{3}$	1	8		
0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	0	4	$\leftarrow R_3+\frac{1}{4}R_2$	
0	1	0	$-\frac{1}{3}$	$\frac{4}{3}$	0	4	$\leftarrow R_4-\frac{1}{4}R_2$	

The advantage of this method is that it can be readily extended to problems with more than two variables, as shown below

Example

Maximise $P = 4x + 5y + 3z$

subject to $8x + 5y + 2z \le 3$

$$3x + 6y + 9z \le 2$$

$$x, y, z \ge 0$$

Solution

As usual slack variables r and s are introduced;

$$8x + 5y + 2z + r = 3$$
$$3x + 6y + 9z + s = 2$$

Now $x,\ y,\ z,\ r,\ s \ge 0$ and the simplex tableau is shown below.

P	***x***	***y***	***z***	***r***	***s***			**Comments**
1	−4	−5	−3	0	0	0		Increase x initially and compare $3/8$*, $2/3$. This smaller positive value of x occurs where $y = z = r = 0$ and so now manipulate the matrix to express P in terms of y, z and r.
0	8	5	2	1	0	3		
0	3	6	9	0	1	2		
1	−4	−5	−3	0	0	0		
0	1	$\frac{5}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0	$\frac{3}{8}$	$\leftarrow R_2/8$	
0	3	6	9	0	1	2		
1	0	$-\frac{5}{2}$	−2	$\frac{1}{2}$	0	$\frac{3}{2}$	$\leftarrow R_1 + 4R_2$	Increase y and compare $(3/8)/(5/8)$, $(7/8)/(33/8)$*. This smaller positive value occurs when $z = r = s = 0$ and so express P in terms of z, r and s.
0	1	$\frac{5}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0	$\frac{3}{8}$		
0	0	$\frac{33}{8}$	$\frac{33}{4}$	$-\frac{3}{8}$	1	$\frac{7}{8}$	$\leftarrow R_3 - 3R_2$	
1	0	$-\frac{5}{2}$	−2	$\frac{1}{2}$	0	$\frac{3}{2}$		
0	1	$\frac{5}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0	$\frac{3}{8}$		
0	0	1	2	$-\frac{1}{11}$	$\frac{8}{33}$	$\frac{7}{33}$	$\leftarrow R_3 / \left(\frac{33}{8}\right)$	
1	0	0	3	$\frac{3}{11}$	$\frac{20}{33}$	$\frac{67}{33}$	$\leftarrow R_1 + \frac{5}{2} R_3$	The first row now has positive coefficients, showing that there is a maximum of $67/33$ when $z = r = s = 0$ (which happens when $x = 8/33$, $y = 7/33$ and $z = 0$).
0	1	0	−1	$\frac{2}{11}$	$-\frac{5}{33}$	$\frac{8}{33}$	$\leftarrow R_2 - \frac{5}{8} R_3$	
0	0	1	2	$-\frac{1}{11}$	$\frac{8}{33}$	$\frac{7}{33}$		

Exercise 5D

Use the simplex algorithm to solve the following problems.

1. Maximise $P = 4x + 6y$

$$\begin{aligned} \text{subject to } \quad x + y &\le 8 \\ 7x + 4y &\le 14 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

2. Maximise $P = 10x + 12y + 8z$

$$\begin{aligned} \text{subject to } \quad 2x + 2y &\le 5 \\ 5x + 3y + 4z &\le 15 \\ x &\ge 0 \\ y &\ge 0 \\ z &\ge 0 \end{aligned}$$

3. Maximise $P = 3x + 8y - 5z$

$$\begin{aligned} \text{subject to } \quad 2x - 3y + z &\le 3 \\ 2x + 5y + 6z &\le 5 \\ x &\ge 0 \\ y &\ge 0 \\ z &\ge 0 \end{aligned}$$

4. Maximise $3x + 6y + 2z$

$$\begin{aligned} \text{subject to } \quad 3x + 4y + 2z &\le 2 \\ x + 3y + 2z &\le 1 \\ x &\ge 0 \\ y &\ge 0 \\ z &\ge 0 \end{aligned}$$

5.5 Miscellaneous Exercises

1. Find the solution to Question 3 of Exercise 5A.

2. A firm manufactures two types of box, each requiring the same amount of material.

   They both go through a folding machine and a stapling machine.

   Type A boxes require 4 seconds on the folding machine and 3 seconds on the stapling machine.

   Type B boxes require 2 seconds on the folding machine and 7 seconds on the stapling machine.

   Each machine is available for 1 hour.

   There is a profit of 40p on Type A boxes and 30p on Type B boxes.

   How many of each type should be made to maximise the profit?

3. A small firm which produces radios employs both skilled workers and apprentices. Its workforce must not exceed 30 people and it must make at least 360 radios per week to satisfy demand. On average a skilled worker can assemble 24 radios and an apprentice 10 radios per week.

   Union regulations state that the number of apprentices must be less than the number of skilled workers but more than half of the number of skilled workers.

   What is the greatest number of skilled workers than can be employed?

   Skilled workers are paid £300 a week, and apprentices £100 a week.

   How many of each should be employed to keep the wage bill as low as possible?

4. In laying out a car park it is decided, in the hope of making the best use of the available parking space (7200 sq.ft.), to have some spaces for small cars, the rest for large cars. For each small space 90 sq.ft. is allowed, for each large space 120 sq.ft. Every car must occupy a space of the appropriate size. It is reliably estimated that, of the cars wishing to park at any given time, the ratio of small to large will be neither less that 2:3 nor greater than 2:1.

   Find the number of spaces of each type in order to maximise the number of cars that can be parked.

*5. A contractor hiring earth moving equipment has the choice of two machines.

   Type A costs £25 per day to hire, needs one man to operate it and moves 30 tonnes of earth per day.

   Type B costs £10 per day to hire, needs four men to operate it and moves 70 tonnes of earth per day.

   The contractor can spend up to £500 per day, has a labour force of 64 men available and can use a maximum of 25 machines on the site.

   Find the maximum weight of earth that the contractor can move in one day.

6. A landscape designer has £200 to spend on planting trees and shrubs to landscape an area of 1000 m^2. For a tree he plans to allow 25 m^2 and for a shrub 10 m^2. Planting a tree will cost £2 and a shrub £5.

   If he plants 30 shrubs what is the maximum number of trees he can plant?

   If he plants 3 shrubs for every tree, what is the maximum number of trees he can plant?

7. A small mine works two coal seams and produces three grades of coal. It costs £10 an hour to work the upper seam, obtaining in that time 1 tonne of anthracite, 5 tonnes of best quality coal and 2 tonnes of ordinary coal. The lower seam is more expensive to work, at a cost of £15 per hour, but it yields in that time 4 tonnes of anthracite, 6 tonnes of best coal and 1 tonne of ordinary coal. Faced with just one order, for 8 tonnes of anthracite, 30 tonnes of best coal and 8 tonnes of ordinary coal, how many hours should each seam be worked so as to fill this order as cheaply as possible?

8. A cycle manufacturer produces two types of mountain-bike: a basic Model X and a Super Model Y. Model X takes 6 man-hours to make per unit, while Model Y takes 10 man-hours per unit. There is a total of 450 man-hours available per week for the manufacture of the two models.

   Due to the difference in demand for the two models, handling and marketing costs are £20 per unit for Model X, but only £10 per unit for Model Y. The total funds available for these purposes are £800 per week.

   Profits per unit for Models X and Y are £20 and £30 respectively. The objective is to maximise weekly profits by optimising the numbers of each model produced.

   (a) The weekly profit is £P. The numbers of units of Model X and Model Y produced each week are x and y. Express P in terms of x and y. Also write down inequalities representing the constraints on production.

   (b) By graphical means or by the simplex method, find the maximum obtainable profit and the numbers of each model manufactured which give this profit.

   (c) If competition forces the manufacturer to give a £5 discount on the price of Model X, resulting in a £5 reduction in profit, how are weekly profits now maximised? (AEB)

9. In order to supplement his daily diet someone wishes to take some Xtravit and some Yeastalife tablets. Their contents of iron, calcium and vitamins (in milligrams per tablet) are shown in the table.

Tablet	Iron	Calcium	Vitamin
Xtravit	6	3	2
Yeastalife	2	3	4

   (a) By taking x tablets of Xtravit and y tablets of Yeastalife the person expects to receive at least 18 milligrams of iron, 21 milligrams of calcium and 16 milligrams of vitamins. Write these conditions down as three inequalities in x and y.

   (b) In a coordinate plane illustrate the region of those points (x,y) which simultaneously satisfy $x \geq 0,\ y \geq 0,$ and the three inequalities in (a).

   (c) If the Xtravit tablets cost 10p each and the Yeastalife tablets cost 5p each, how many tablets of each should the person take in order to satisfy the above requirements at the minimum cost? (AEB)

10. A maker of wooden furniture can produce three different types of furniture: sideboards, tables and chairs. Two machines are used in the production - a jigsaw and a lathe.

    The manufacture of a sideboard requires 1 hour on the jigsaw and 2 hours on the lathe; a table requires 4 hours on the jigsaw and none on the lathe; a chair requires 2 hours on the jigsaw and 8 hours on the lathe.

    The jigsaw can only operate 100 hours per week and the lathe for 40 hours per week. The profit made on a sideboard is £100, £40 on a table and £10 on a chair. In order to determine how best to use the two machines so as to maximise profits, formulate the problem as a linear programming problem, and solve it using the simplex tableau.

11. A diet-conscious housewife wishes to ensure her family's daily intake of vitamins A, B and C does not fall below certain levels, say 24 units, 30 units and 18 units, respectively. For this she relies on two fresh foods which, respectively, provide 8, 5 and 2 units of vitamins per ounce of foodstuff and 3, 6 and 9 units per ounce. If the first foodstuff costs 3p per ounce and the second only 2p per ounce, use a graphical method to find how many ounces of each foodstuff should be bought by the housewife daily in order to keep her food bill as low as possible.