4 INEQUALITIES

Objectives

After studying this chapter you should

- be able to manipulate simple inequalities;
- be able to identify regions defined by inequality constraints;
- be able to use arithmetic and geometric means;
- be able to use inequalities in problem solving.

4.0 Introduction

Since the origin of mankind the concept of one quantity being greater than, equal to or less than, another must have been present. Human greed and 'survival of the fittest' imply an understanding of inequality, and even as long ago as 250 BC, Archimedes was able to state the inequality

$$3\frac{10}{71} < \pi < 3\frac{10}{70}.$$

Nowadays we tend to take inequalities for granted, but the concept of **inequality** is just as fundamental as that of **equality**.

You certainly meet inequalities throughout life, though often without too much thought. For example, in the United Kingdom the temperature $T°C$ is usually in the range

$$-15 < T < 30$$

and it would be extremely cold or hot if the temperature was outside this range. In fact, animal life can exist only in the narrow band of temperature defined by

$$-60 < T < 60.$$

It will be assumed that you are familiar with a basic understanding of the use of inequalities $<$, $\leq$, $>$ and $\geq$, and that you have already met the graphical illustration of simple inequalities. You will cover this ground again, but experience with this using Cartesian coordinates and some competence in algebraic manipulation would be very helpful.

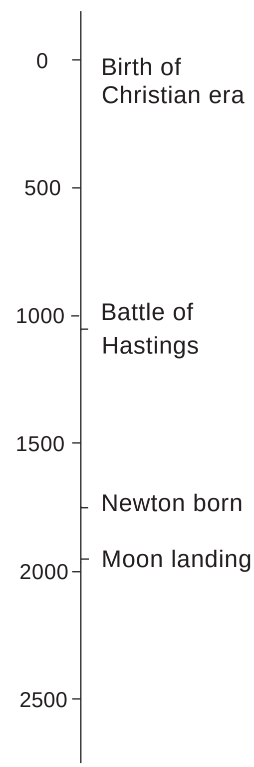
Activity 1

Find and prove an inequality relationship for π.

4.1 Fundamentals

The concept of 'greater than' or 'less than' enables numbers to be ordered, and represented on, for example, a number line.

The time line opposite gives a time scale for some important events.



You can also use inequalities for other quantities. For example, the speed of a small car will normally lie within the limits

$$-15\text{mph} < \text{speed} < 120\text{mph}.$$

Before looking at more inequality relationships the definition must be clarified. Writing $x > y$ simply means that $x - y$ is a positive number: the other inequalities $<$, $\geq$, $\leq$ can be defined in a similar way. Using this definition, together with the fact that the sum, product and quotient of two positive numbers are all positive, you can prove various inequality relationships.

Example

Show that

(a) if $u > v$ and $x > y$ then $u + x > v + y$;

(b) if $x > y$ and k is a positive number, then $kx > ky$.

Solution

(a) If $u > v$ and $x > y$ then this simply means that $u - v$ and $x - y$ are both positive numbers: hence their sum

$$u - v \text{ and } x - y$$

is also positive. But this can be rewritten as

$$(u + x) - (v + y).$$

Since this difference is a positive number you can deduce that

$$u + x > v + y$$

as required.

(b) If $x > y$ then this simply means that $x - y$ is a positive number. Since k is also positive you can deduce that the product $k(x - y)$ is positive. Therefore

$$kx - ky = k(x - y)$$

is a positive number, which means that $kx > ky$ as required.

Activity 2

What happens to property (b) when k is a **negative** number?

What happens to subtraction of inequalities? For example, if $u > v$ and $x > y$ then is it always true that $u - v > x - y$?

Can you take square roots through an inequality? i.e. If $a^2 > b^2$ then is it necessarily true that $a > b$?

Investigate these questions with simple illustrations.

In what follows you will need to solve and interpret inequalities. These inequalities will usually be **linear** (that means not involving powers of x, etc), but you will first see how to solve more complex inequalities. The procedure is illustrated in the following example.

Example

Find the values of x which satisfy the inequality

$$x^2 + 7 < 3x + 5.$$

Solution

You can rewrite the inequality as

$$x^2 + 7 - (3x + 5) < 0$$

$$x^2 - 3x + 2 < 0$$

$$(x - 2)(x - 1) < 0.$$

Since the complete expression is required to be negative, this means that one bracket must be positive and the other negative.

This will be the case when $1 < x < 2$.

Exercise 4A

1. Prove that if $x > y > 0$, then $\frac{1}{y} > \frac{1}{x}$.

2. Prove that if $a^2 > b^2$, where a and b are positive numbers, then $a > b$.

3. Find the values of x for which $8 - x \geq 5x - 4$.

4. Find in each case the set of real values of x for which

   (a) $3(x-1) \geq x+1$

   * (b) $\frac{3}{(x-1)} \geq \frac{1}{(x+1)}$

5. Find the set of values of x for which

$$x^2 - 5x + 6 \geq 2.$$

4.2 Graphs of inequalities

In the last section it was shown that inequalities can be solved algebraically; however, it is often more instructive to use a graphical approach.

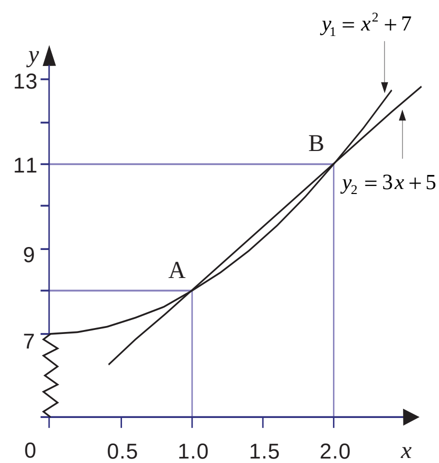
Consider the previous example in which you want to find values of x which satisfy

$$x^2 + 7 < 3x + 5.$$

Another approach is to draw the graphs of

$$y_1 = x^2 + 7, \quad y_2 = 3x + 5$$

and note when $y_1 < y_2$. This is illustrated in the graph opposite.



Between the points of intersection, A and B, $y_2 > y_1$. Solving the equation $y_1 = y_2$ gives

$$x^2 + 7 = 3x + 5$$

$$\Rightarrow \quad x^2 - 3x + 2 = 0$$

$$\Rightarrow \quad (x-2)(x-1) = 0$$

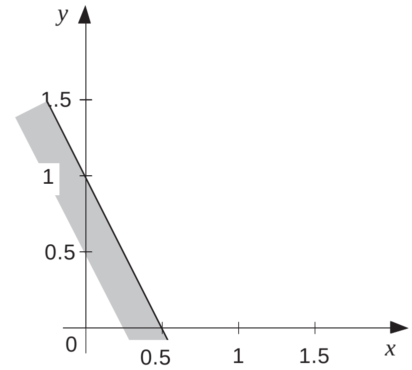
$$\Rightarrow \quad x = 1 \text{ or } 2$$

giving, as before, the solution $1 < x < 2$.

You will find a graphical approach particularly helpful when dealing with inequalities in two variables.

Example

Find the region which satisfies $2x + y > 1$.



Solution

The boundary of the required region is found by solving the **equality**

$$2x + y = 1.$$

This is shown in the diagram opposite

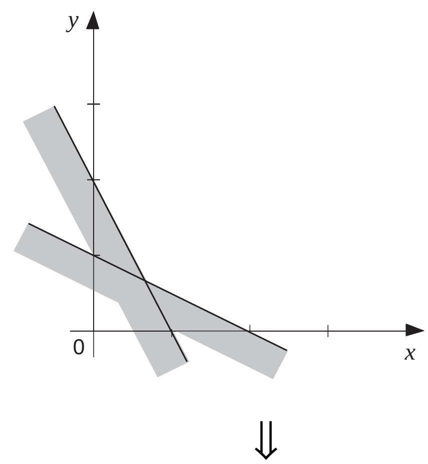
The inequality will be satisfied by all points on one side of the line. To identify which side, you can test the point (0, 0) - this does not satisfy the inequality, so the region to the right of the line is the solution. The **excluded** region is on the **shaded** side of the line.

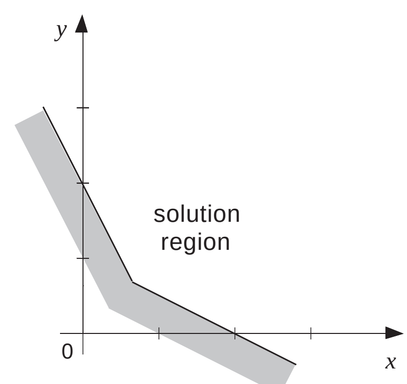
Just as you can solve simultaneous equations, you can tackle simultaneous inequalities. For example, suppose you require values of x and y which satisfy

$$2x + y > 1$$

and $$x + 2y > 1.$$

You have already solved the first inequality, and if you add on the graph of the second inequality, you obtain the region as shown in this diagram.



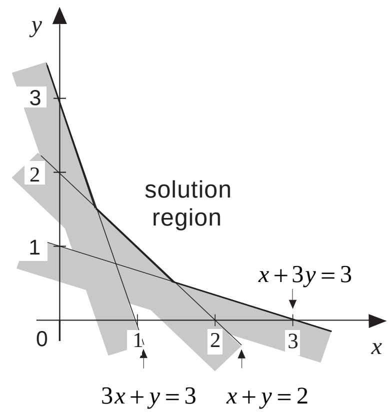


Combining the two inequalities gives the solution region as shown opposite.

Example

Find the region which satisfies all of the following inequalities.

$$x+y>2$$
$$3x+y>3$$
$$x+3y>3$$

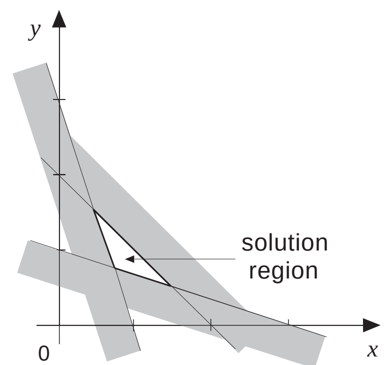


Solution

As before, the graph of the three inequalities is first drawn and the region in which **all** three are satisfied is noted.

Note that if you had wanted to solve

$$x+y<2$$
$$3x+y>3$$
$$x+3y>3$$



then the solution would have been the triangular region completely bounded by the three lines; in general the word **finite** will be used for such bounded regions.

Activity 3

Write down **three** different linear equations of the form

$$ax+by=c.$$

Which three inequalities are satisfied in the finite region formed by these lines?

Suppose you now have **four** linear inequalities in x and y to be satisfied.

What regions might they define?

The following example illustrates some of the possibilities.

Example

In each case find the solution region.

(a) $x+y>1,\quad y-x<1,\quad 2y-x>0,\quad 2x+3y<6$

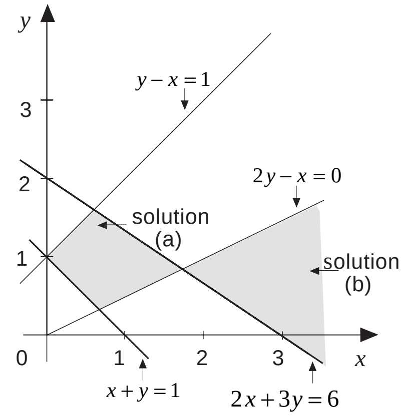
(b) $x+y>1,\quad y-x<1,\quad 2y-x<0,\quad 2x+3y>6$

(c) $x+y<1,\quad y-x>1,\quad 2y-x<0,\quad 2x+3y>6$

Solution

First graph $x+y=1,\ y-x=1,\ 2y-x=0$ and $2x+3y=6,$ and then in each case identify the appropriate region.

There is no region which satisfies (c).



How many finite regions are formed by the intersection of four lines?

Exercise 4B

1. Solve graphically the inequalities

   (a) $(2-3x)(1+x)\le 0$

   (b) $x^2 \le 2x+8.$

2. Solve

   $y\ge 0,\ x+y\le 2$ and $y-2x<2.$

3. Find the solution set for

   $x+y<1$ and $3x+2y<6.$

4. Find the region which satisfies

   $x+y\ge 2$
   $x+4y\le 4$
   $y>-1.$

5. Is the region satisfying
   $x+y>1,\ 3x+2y<12,\ y-x<2,\ 2y-x>1$
   finite?

$2x+3y=6$

4.3 Classical inequalities

You are probably familiar with the arithmetic mean (often called the average) of a set of positive numbers. The **arithmetic mean** is defined for positive numbers $x_1,\ x_2,\ \dots,\ x_n$ by

$$A = \frac{x_1+x_2+\ \dots\ +x_n}{n}$$

So, for example, if $x_1=5,\ x_2=6,\ x_3=10,$ then

$$A = \frac{(5+6+10)}{3} = 7.$$

There are many other ways of defining a mean; for example, the **geometric mean** is defined as

$$G = (x_1 x_2\ \dots\ x_n)^{1/n}$$

For the previous example,

$$G = (5 \times 6 \times 10)^{1/3} = (300)^{1/3} \approx 6.69$$

The **harmonic mean** is defined by

$$\boxed{\frac{1}{H} = \frac{1}{n}\left(\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}\right)}$$

So, again with $x_1 = 5,\ x_2 = 6,$ and $x_3 = 10,$

$$\frac{1}{H} = \frac{1}{3}\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{10}\right) = \frac{1}{3} \times \frac{7}{15}$$

giving

$$H = \frac{45}{7} \approx 6.43$$

Activity 4

For varying positive numbers $x_1, x_2, x_3,$ find the arithmetic, geometric and harmonic means. What inequality can you conjecture which relates to these three means?

If you have tried a variety of data in Activity 4, you will have realised that the geometric and harmonic means give less emphasis to more extreme numbers. For example, given the numbers 1, 5 and 9,

$$A = 5,\ G = 3.56,\ H = 2.29,$$

whereas for numbers 1, 5 and 102,

$$A = 36,\ G = 7.99,\ H = 2.48.$$

Whilst the arithmetic mean has changed from 5 to 36, the geometric mean has only doubled, and the harmonic mean has hardly changed at all!

In most calculations for mean values the **arithmetic mean** is used, but not always.

One criterion which any mean must satisfy is that, when all the numbers are equal, i.e. when $x_1 = x_2 = \ldots = x_n (= a)$ say, then the mean must equal a.

For example,

$$A = \frac{a + a + \dots + a}{n} = \frac{na}{n} = a$$

$$G = (a\,a\,a\,\dots\,a)^{1/n} = \left(a^n\right)^{1/n} = a.$$

Similarly, $H = a$ when all the numbers are equal.

Activity 5

Define a new mean of n positive numbers $x_1, x_2, \dots, x_n$ and investigate its properties.

In Activity 4 you might have realised that

$$A \ge G \ge H$$

(equality only occurring when all the numbers are equal). The first inequality will be proved for any two positive numbers, x_1 and x_2.

Given the inequality

$$\left(x_1 - x_2\right)^2 \ge 0$$

then equality can occur only when $x_1 = x_2$.

This inequality can be rewritten as

$$x_1^2 - 2x_1x_2 + x_2^2 \ge 0$$

or $$x_1^2 + 2x_1x_2 + x_2^2 \ge 4x_1x_2 \text{ (adding } 4x_1x_2 \text{ to each side)}$$

giving $$\frac{\left(x_1 + x_2\right)^2}{4} \ge x_1x_2.$$

Taking the positive square root of both sides, which was justified in Question 2 of Exercise 4A,

$$\boxed{\frac{x_1 + x_2}{2} \ge \sqrt{x_1x_2}}$$

i.e. $A \ge G$

and equality only occurs when $x_1 = x_2$.

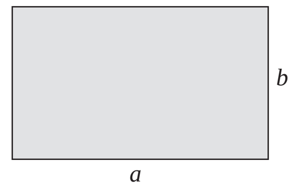
You will see how this result can be used in geometrical problems.

Example

Show that of all rectangles having a given perimeter, the square encloses the greatest area.

Solution

For a rectangle of sides a and b, the perimeter, L is given by



$$L = 2(a+b),$$

and the area, A, by

$$A = ab.$$

Using the result above, with x_1 replaced by a and x_2 replaced by b,

$$\frac{L}{4} \geq \sqrt{A}$$

or $$A \leq \frac{L^2}{16}$$

where equality occurs only when $a = b$. Since in this example the perimeter is fixed, the right hand side of this last inequality is constant: also equality holds if and only if $a = b$. Therefore you can deduce that the maximum value of A is $\frac{L^2}{16}$ and that it is only obtained for the square.

In fact, the inequalities

$$A \geq G \geq H$$

hold for any set of positive numbers, $x_1, x_2, \ldots, x_n$, but the result is not easy to prove, and requires, for example, the use of a mathematical process called induction.

The result in the example above illustrates what is called an **isoperimetric inequality**; you will see more of these in the next section.

Exercise 4C

1. Find the arithmetic, geometric and harmonic means for the following sets of numbers, and check that the inequality $A \geq G \geq H$ holds in each case.

   (a) 1, 2, 3, 4;

   (b) 0.1, 2, 3, 4.9;

   (c) 0.1, 2, 3, 100;

   (d) 0.001, 2, 3, 1000;

   (e) 0.001, 0.002, 1000, 2000.

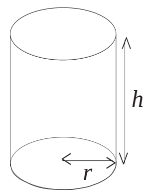
2. Prove that $G \geq H$ for any two positive numbers x_1 and x_2.

*3. By taking functions $\frac{1}{x^2}$ and x^2 as numbers in the arithmetic/geometric mean inequality, find the least value of

$$y = \frac{1+x^4}{x^2}.$$

*4. Show that the surface area, S, of a closed cylinder of volume V can be written as

$$S = 2\pi r^2 + \frac{2V}{r}.$$



Writing

$$S = 2\pi\left(r^2 + \frac{2V}{2\pi r}\right)$$

and using the arithmetic and geometric means inequality for the three numbers

$$r^2,\ \frac{V}{2\pi r},\ \frac{V}{2\pi r}$$

show that

$$\frac{S}{6\pi} \geq \left(\frac{V^2}{4\pi^2}\right)^{\frac{1}{3}}.$$

When does equality occur? What relationship does this give between h and r?

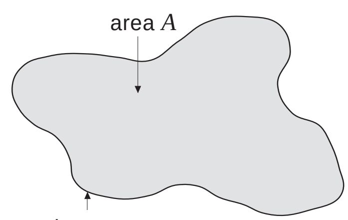
4.4 Isoperimetric inequalities

In the last section there was an example of an isoperimetric inequality. You will look at a more general result (first known to the Greeks in about 2000 BC) and at some further special cases.

According to legend, Princess Dido was fleeing from the tyranny of her brother and, with her followers, set sail from Greece across the sea. Having arrived at Carthage, she managed to obtain a grudging concession from the local native chief to the effect that

> 'she could have as much land as could be encompassed by an ox's skin.'

Of course, the natives expected her to kill the biggest ox she could find and use its skin to claim her land - but her followers were very astute, advising her to cut the skin to make as many thin strands as possible and to join them together to form one long length to mark the perimeter of her land. Her only problem then was in deciding what shape this perimeter should be to enclose the maximum area.



What do you think is the best shape?

In mathematical terms, the search is for the shape which maximises the area A inside a given perimeter of length L.

Example

For a given perimeter length, say 12 cm, find the area enclosed by

(a) a square;

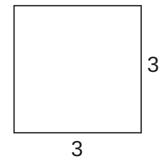
(b) a circle;

(c) an equilateral triangle.

Solution

(a) For $L = 12$, each side is of length 3 cm

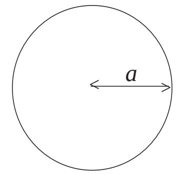
and $$A = 3^2 = 9 \text{ cm}^2.$$



(b) For $L = 12$, assume the radius is *a*, giving

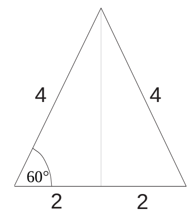
$$12 = 2\pi a \Rightarrow a = \frac{6}{\pi}$$

and $$A = \pi a^2 = \pi \left(\frac{6}{\pi}\right)^2 = \frac{36}{\pi} \approx 11.46 \text{ cm}^2$$

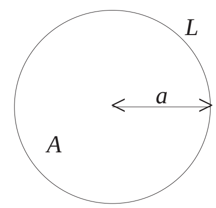


(c) Again, for $L = 12$, each side is of length 4 cm, and

$$A = \frac{1}{2} \times 4 \times 4 \sin 60 = 4\sqrt{3} \approx 6.93 \text{ cm}^2.$$



So, for the particular problem of a perimeter length of 12 cm, of the three shapes chosen the circle gives the largest area - but can there be another shape which gives a larger one? You can make some progress by looking more carefully at the circle in the general case of perimeter L.



Now $$L = 2\pi a$$

and $$A = \pi a^2 = \pi \left(\frac{L}{2\pi}\right)^2 = \frac{L^2}{4\pi}$$

So, for **any** circle

$$\frac{4\pi A}{L^2} = 1.$$

The **Isoperimetric Quotient Number** (I.Q.) of any closed curve is defined as

$$\boxed{\text{I. Q.} = \frac{4\pi A}{L^2}}$$

For the circle, you see that I.Q. = 1. In the basic problem you have been trying to find the shape which gives a maximum value to *A* for a fixed value of L. In terms of the I.Q. number, you want to find the shape which gives the maximum value to the I.Q. number. But, for a circle, the value of the I.Q. number is 1, so if the optimum shape is a circle, then the inequality

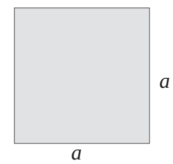
$$\boxed{\text{I.Q.} \leq 1}$$

holds for all plane shapes, and equality occurs **only** for the circle.

Note that, since the I.Q. is the ratio of an area to the square of a length, it is non-dimensional, i.e. a number requiring no units.

Example

Find the I.Q. number for a square of side a.



Solution

$$L = 4a, \; A = a^2,$$

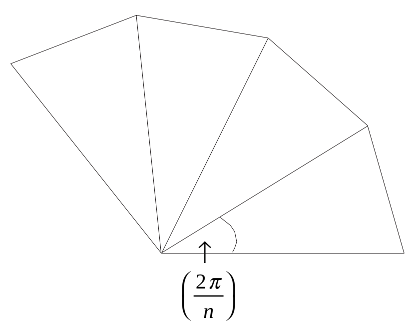
and $$\text{I.Q.} = 4\pi \times \frac{a^2}{(4a)^2} = \frac{\pi}{4} \approx 0.785.$$

Activity 6

Find I.Q. numbers of various shapes and check that, in each case, the inequality I.Q. ≤ 1 holds.

A complete proof is beyond the scope of this present work (and, in fact, involves high level mathematics). It is surprising that such a simple result, known to the Greeks, could not be proved until the late 19th Century, and even then required sophisticated mathematics. You can, though, verify the result for all regular polygons as will be shown.

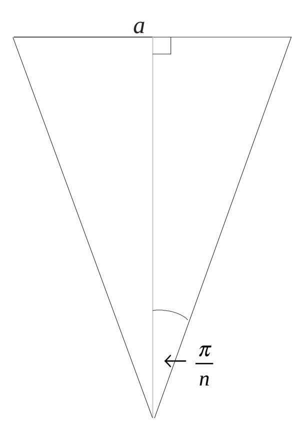
Consider a regular polygon of n sides. The angle subtended by each side at the centre is



$$\frac{360}{n} \text{ degrees or } \frac{2\pi}{n} \text{ radians.}$$

You will work in **radians** in what follows. If each side is of length a, the area of each triangle is given by

$$\frac{1}{2} \times a \times \frac{a}{2} \times \frac{1}{\tan\left(\frac{\pi}{n}\right)} = \frac{a^2}{4\tan\left(\frac{\pi}{n}\right)}$$



The total area, $A = \dfrac{na^2}{4\tan\left(\frac{\pi}{n}\right)}$, and $L = na$,

so $$\text{I.Q.} = 4\pi \times \left(\frac{na^2}{4\tan\left(\frac{\pi}{n}\right)}\right) \times \frac{1}{(na)^2}$$

i.e. $$\text{I.Q.} = \frac{\left(\frac{\pi}{n}\right)}{\tan\left(\frac{\pi}{n}\right)}$$

Activity 7

Use your calculator to find the limit of $\left(\dfrac{x}{\tan x}\right)$ as $x \to 0$.

Now you can write I.Q.$= \dfrac{x}{\tan x}$ where $x = \dfrac{\pi}{n}$.

As $n \to \infty$, the polygon becomes, in the limit, a circle, and you have seen that I.Q.$=1$, as expected. Note that, for all values of the positive integer n except $n=1$, $\tan\left(\dfrac{\pi}{n}\right) > \left(\dfrac{\pi}{n}\right)$ - use your calculator to check some of the values. Hence, for any regular polygon

$$\text{I.Q.} \leq 1,$$

and you can see that the larger n becomes, the closer the I.Q. comes to 1, fitting in with the fact that the I.Q. for a circle is 1.

Finally, it should be noted that Princess Dido did not live happily ever after. Having been outwitted by her, the native leader promptly fell in love with her. As she did not reciprocate his feelings, she burnt herself on a funeral pyre in order to escape a fate worse than death!

Exercise 4D

1. For a given perimeter length of 12 cm, find the total area enclosed by the rectangle with sides
   (a) 3 cm and 3 cm
   (b) 2 cm and 4 cm
   (c) 1 cm and 5 cm
2. Find the I.Q. numbers for the following shapes:
   (a) equilateral triangle;
   (b) regular hexagon;
   (c) rectangle with sides in the ratio 1 : 2.
3. For a rectangle with sides in the ratio
   $1:k \quad (k \leq 1)$
   find an expression for the I.Q. number. What value of k gives:
   (a) maximum value
   (b) minimum value
   for the I.Q. number?
4. What is the volume, V, of the sphere which is enclosed by a surface area of 12 cm^2?
5. What is the volume, V, of the cube which is enclosed by a surface area of 12 cm^2?
6. For a given surface area, S, what closed three-dimensional shape do you think gives a maximum volume?

4.5 Miscellaneous Exercises

1. Obtain the sets of values of x for which

   (a) $2x > \frac{1}{x}$

   (b) $\frac{1}{x+1} > \frac{x}{3+x}$.

2. Find the range of values of x for which

   $4x^2 - 12x + 5 < 0$.

3. Find the ranges of values of x such that

   $x > \frac{2}{x-1}$.

4. Find the set of values of x for which

   $\frac{x(x+2)}{x-3} < x+1$.

5. Find the solution set of the pair of inequalities

   $x + y < 1$
   $2x + 5y < 10$.

6. Is the region defined by

   $2x - 3y \le 6$
   $x + y \le 4$

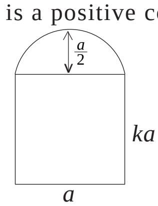
   finite?

7. Find the region satisfied by

   $x + y \le 4$
   $2x - 3y \le 6$
   $3x - y \ge -3$
   $x \le 2$.

8. Prove that $A \ge H$ for any two positive numbers. When does equality occur?

9. Find the I.Q. number for the shape illustrated below, where k is a positive constant.



   What value of k gives a maximum I.Q. value?

10. Find the I.Q. number for a variety of triangles, including an equilateral triangle. What do you deduce about the I.Q. numbers for triangles?

*11. For three-dimensional closed shapes, the isosurface area quotient number is defined as

$$\text{I.Q.} = \frac{6\sqrt{\pi}\,V}{S^{\frac{3}{2}}}$$

   where V is the volume enclosed by a total surface area S. Find the I.Q. for a variety of three-dimensional shapes. Can you find an inequality satisfied by all closed shapes in three dimensions?