15 Variation

15.1 Simple Ratios

Ratios are used in many situations to describe how two quantities are related. For example, a cake recipe requires twice as much flour as margarine. The ratio of flour to margarine is 2 : 1. Sometimes ratios can be simplified; for example, the ratio 100 : 50 is the same as the ratio 2 : 1. Ratios are simplified in a very similar way to fractions.

Worked Example 1

Simplify each of the following ratios.

(a) 48 : 12 (b) 27 : 9 (c) 35 : 49

Solution

(a) Both numbers in the ratio can be divided by 12. This gives

$$48 : 12 = 4 : 1$$

Alternatively, the ratio can be simplified in a number of steps.

$$\begin{aligned} 48 : 12 &= 24 : 6 \\ &= 12 : 3 \\ &= 4 : 1 \end{aligned}$$

(b) Here both numbers in the ratio can be divided by 9. This gives

$$27 : 9 = 3 : 1$$

Alternatively, the ratio can be simplified in steps to give

$$\begin{aligned} 27 : 9 &= 9 : 3 \\ &= 3 : 1 \end{aligned}$$

(c) Here both numbers can be divided by 7 to give

$$35 : 49 = 5 : 7$$

Worked Example 2

A school class contains 12 girls and 20 boys. Find the ratio of:

(a) girls to boys,

(b) boys to girls.

Solution

(a) The ratio of girls to boys is:

$$12 \text{ to } 20 \text{ or } 12 : 20$$

This can be simplified by dividing both numbers by 4, to give

$$12 : 20 = 3 : 5$$

(b) To find the ratio of boys to girls, reverse the ratio of girls to boys, to give 5 : 3.

Worked Example 3

A glass contains 300 cm^3 of drink. The drink is made by mixing 50 cm^3 of concentrate with water. Find the ratio of concentrate to water.

Solution

$$\begin{aligned}\text{Amount of water} &= 300 - 50 \\ &= 250 \text{ cm}^3\end{aligned}$$

The ratio of concentrate to water is

$$50 : 250$$

which simplifies to

$$1 : 5$$

Worked Example 4

The ratio of blue Smarties to other coloured Smarties in one packet is 1 : 12. How many Smarties would there be in the packet if it contained:

(a) 3 blue Smarties?

(b) 5 blue Smarties?

Solution

The ratio of 1 : 12 means that for every blue Smartie there are 12 Smarties of other colours.

(a) This packet contains 3 blue Smarties and $3 \times 12 = 36$ other Smarties.

In total the packet contains $3 + 36 = 39$ Smarties.

(b) This packet contains 5 blue Smarties and $5 \times 12 = 60$ other Smarties. This give a total of 65 Smarties.

Exercises

1. Simplify each of the following ratios.

(a) 4 : 2	(b) 8 : 2	(c) 3 : 6
(d) 9 : 12	(e) 5 : 30	(f) 8 : 42
(g) 3 : 18	(h) 25 : 75	(i) 8 : 100
(j) 30 : 240	(k) 64 : 80	(l) 21 : 15
(m) 81 : 48	(n) 32 : 100	(o) 50 : 49
(p) 4.8 : 1.2	(q) 10.5 : 3.5	(r) 8.6 : 30.1

2. A school contains 300 girls and 320 boys. Find:

   (a) the ratio of girls to boys,

   (b) the ratio of boys to girls.

3. The shape of a room is a rectangle with sides of length 5 m and 3.5 m. Find the ratio of:

   (a) the length to the width,

   (b) the width to the length.

4. Two different bus companies have different pricing policies.

   For Company A on one route the adult fare is £1.20 and the child fare is 40p.

   For Company B on a different route, the adult fare is £1.40 and the child fare is 70p.

   (a) Find the ratio of the child fare to the adult fare for each company.

   (b) Which company gives children the better deal?

5. A library contains 720 non-fiction books and 400 fiction books.

   (a) Find the ratio of fiction books to non-fiction books.

   (b) Find the new ratio if 40 new fiction books are bought for the library.

6. A fizzy drink contains lemonade and 60 cm^3 of orange juice. Find the ratio of juice to lemonade in:

   (a) a 100 cm^3 drink,

   (b) a 300 cm^3 drink.

7. A car park contains 400 parking spaces. Of these spaces, 60 are short term and the rest long term. Find the ratio of short term spaces to long term spaces.

8. In a season a football team played 60 matches. They won 18, lost 20 and the rest were draws. Find the following ratios:

   (a) number of matches won to number of matches drawn,

   (b) number of matches won to other matches,

   (c) number of matches lost to number of matches won.

9. Orange squash is mixed with water in the ratio 1 : 8, i.e. 1 part orange squash to 8 parts water. How much water is mixed with:

   (a) 100 cm^3 of squash,

   (b) 20 cm^3 of squash,

   (c) 5 cm^3 of squash?

10. In a school the ratio of teachers to pupils is 1 : 20. If there are 12 teachers, how many pupils are there in the school?

11. In a class the ratio of left-handed pupils to right-handed pupils is 1 : 12. There are 2 left-handed pupils in the class. How many pupils are there in the class?

12. In packets of sweets, chocolate covered peanuts are mixed with solid chocolate sweets in the ratio 1 : 3. How many sweets are there in a packet that contains 20 chocolate covered peanuts?

13. In a herd of cattle, the ratio of bulls to cows is 2 : 25. How many cattle would be in the herd of it contained 10 bulls?

15.2 Proportion and Ratio

When solving problems that contain ratios like 4 : 5 or 3 : 7, it is often useful to write the ratios in the form $n : 1$ or $1 : m$.

So $\quad 4 : 5$ is equivalent to $1 : \frac{5}{4}$ or $\frac{4}{5} : 1$

and $\quad 3 : 7$ is equivalent to $1 : \frac{7}{3}$ or $\frac{3}{7} : 1$.

Worked Example 1

In a fruit drink, orange juice and pineapple juice are mixed in the ratio 3 : 7. Find how much pineapple juice would be mixed with 500 cm^3 of orange juice.

Solution

The ratio of orange juice to pineapple juice is

$$3 : 7 \text{ or } 1 : \frac{7}{3}$$

So for every 1 cm^3 of orange juice, $\frac{7}{3}$ cm^3 of pineapple juice is needed.

For 500 cm^3 of orange juice, the amount of pineapple juice needed is

$$500 \times \frac{7}{3} = 1166\frac{2}{3}\ \text{cm}^3.$$

Worked Example 2

Jason buys 8 m of wire netting to make a rabbit run. This costs him £5.04.
Ramada is also making a rabbit run. He needs 6.8 m of wire netting. How much will this cost?

Solution

The cost per metre of the wire netting is:

$$\frac{£5.04}{8} = £0.63 \text{ or 63p per m.}$$

Ramada needs 6.8 m. This will cost

$$6.8 \times 63\text{p} = 428.4\text{p}$$
$$= £4.28 \quad \text{(to the nearest penny)}$$

This is equivalent to using the ratio 6.8 : 8; so solving this problem as a ratio, you take

$$\frac{6.8}{8} \times £5.04 = £4.28, \text{ to the nearest penny (as before).}$$

Worked Example 3

Three men are digging trenches to install cables to connect computers to the internet. They can dig 12 m of trench each day.

(a) How long will it take 2 men to dig 120 m of trench?

(b) How many men will be needed to dig 80 m of trench in 2 days?

Solution

As 3 men dig 12 m each day, 1 man digs 4 m each day.

(a) As each man digs 4 m per day, 2 men dig 8 m per day. The time taken to dig 120 m is given by

$$\frac{120}{8} = 15 \text{ days}$$

Alternatively, 2 men can dig

$$\frac{2}{3} \times 12 = 8 \text{ m}$$

of trench each day, so it will take

$$\frac{120}{8} = 15 \text{ days}$$

to dig the trench.

(b) As each man digs 4 m per day, each man digs 8 m in 2 days, The number of men needed is given by

$$\frac{80}{8} = 10 \text{ men}.$$

Exercises

1. The ratio of the length to width of a photograph is 3 : 2. The photograph is enlarged so that the length is 9 inches.

   What is the width of the enlarged photograph?

2. The ratio of flour to sugar in a recipe is 5 : 4.

   (a) How much flour should be mixed with 100 grams of sugar?

   (b) How much sugar should be mixed with 200 grams of flour?

3. The manager of a music store estimates that the ratio of sales of cassettes to CDs is 4 : 7.

   (a) In one day, 92 cassettes were sold. How many CDs were sold that day?

   (b) On another day, 84 CDs were sold. How many cassettes were sold that day?

4. A 6 m length of rope costs £2.70. Find the cost of the following lengths of rope.

   (a) 1 m (b) 15 m (c) 22 m

5. A greengrocer sells potatoes by the kilogram. She sells 5 kg of potatoes for 60p. Find the cost of:

   (a) 7 kg of potatoes,

   (b) 20 kg of potatoes,

   (c) 2 kg of potatoes.

6. Ribbon is sold from a roll. Rachel buys 3 m for £1.71. Find the cost of each length of ribbon below.

   (a) 5 m (b) 12 m (c) 70 cm

7. A recipe uses 200 grams of flour to make 8 cakes.

   (a) How much flour would be needed for 20 cakes?

   (b) How many cakes could be made with 325 grams of flour?

8. Ben bought 12 litres of petrol for £7.08.

   (a) How much would 40 litres of petrol cost?

   (b) How much petrol could be bought for £10?

9. Shakeel adds the following spices to a recipe for 4 people.

   2 *teaspoons turmeric*

   1 *teaspoons coriander*

   $\frac{1}{2}$ *teaspoon hot chilli powder*

   How much of each ingredient should he use for a recipe for:

   (a) 8 people, (b) 10 people, (c) 3 people?

10. Mr Mosafeer employs 5 men who together can build a wall 12 m long in 3 days.

    (a) How long would it take the men to build a wall 20 m long?

    (b) How long would it take 3 men to build the 12 m wall?

    (c) If the 12 m wall must be built in 2 days, how many more men must Mr Mosafeer employ?

11. A school with 700 pupils employs 25 teachers. Use this ratio of teachers and pupils to answer the following questions.

    (a) How many teachers are needed for a school of 560 pupils?

    (b) How many pupils are there in a school with 22 teachers?

12. Two people can unload a lorry containing 200 boxes in 5 hours.

    (a) If four people are to unload the lorry instead of 2, how much time will be saved?

    (b) A larger lorry contains more boxes. If 6 people can unload it in $2\frac{1}{2}$ hours, how many boxes does it contain?

13. Three photocopiers can produce 5400 copies in one hour.

    (a) How long would it take one photocopier to make 16 200 copies?

    (b) How long would it take two photocopiers to make 1200 copies?

    (c) How many photocopiers would be needed to make 3600 copies in 30 minutes?

14. A decorator find that 2 people can paint 80 m^2 per day. A new warehouse has 560 m^2 of walls to be painted. The decorator wants to complete the work in a number of whole days and does not want to employ more than 10 people to work on the job. How can the work be completed in the shortest time?

15.

Two pints of milk cost 58p.

What is the cost of 5 pints of milk at the same price per pint?

(SEG)

16. Here is a list of ingredients for making some Greek food.

    These amount make enough for 6 people.

    2 cloves of garlic.

    4 ounces of chick peas.

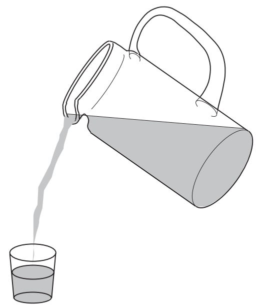
    4 tablespoons of olive oil.

    5 fluid ounces of Tahina paste.

    Change the amount so that there will be enough for 9 people.

(LON)

17. (a)



An orange drink is made by mixing water with concentrated orange juice.

$\frac{3}{4}$ of the orange drink is water.

How many litres of water will be in 12 litres of orange drink?

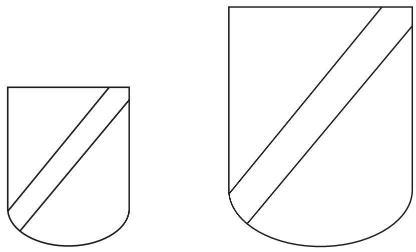
(b) It takes 100 g of flour to make 15 shortbread biscuits.

(i) Calculate the weight of flour needed to make 24 shortbread biscuits.

(ii) How many shortbread biscuits can be made from 1 kg of flour?

(MEG)

18. A school badge is made in two sizes.



Not to scale

The width of the small size is 3 cm.

The large size is an enlargement of the small size in the ratio 2 : 3.

Calculate the width of the large size badge.

(SEG)

19. (a) A shopkeeper orders one hundred 60 g packets of peanuts.

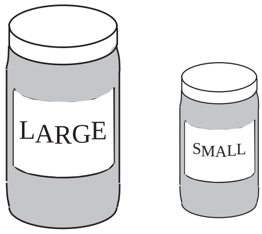
What is the total weight of the order in kilograms?

(b) A 60 g packet of peanuts costs 48 pence.

Calculate the cost of an 80 g packet of peanuts at the same price per gram.

(SEG)

20. Jam is sold in two sizes.



A large pot of jam costs 88p and weighs 822 g.

A small pot of jam costs 47p and weighs 454 g.

Which pot of jam is better value for money?

You **must** show all your working.

(SEG)

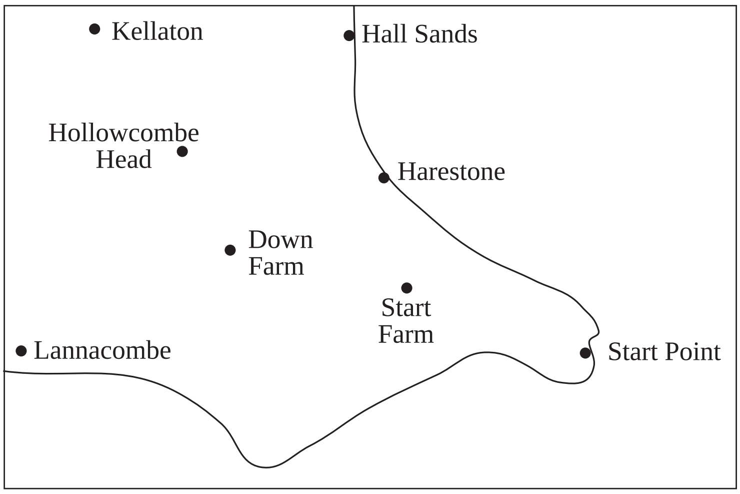
15.3 Map Scales and Ratios

Maps have a scale which relates the distance on the map to the corresponding distance on the ground. For example, on some ordnance survey maps the scale 1 : 50 000 is quoted on the cover. This means that a distance of 1 cm on the map corresponds to 50 000 cm, or 500 m, on the ground.

1 : 50 000

Worked Example 1

The map shown below has been drawn with a scale of 1 : 50 000.



Find the distances between:

(a) Down Farm and Start Farm,

(b) Kellaton and Harestone.

Give your answer in kilometres.

Solution

As the scale is 1 : 50 000, each centimetre on the map represents 50 000 cm in reality.

(a) From the map the distance between Down Farm and Start Farm can be measured as 2.4 cm.

The actual distance between the two points is given by:

$$\begin{aligned} 2.4 \times 50\,000 &= 120\,000 \text{ cm} \\ &= 1200 \text{ m} \\ &= 1.2 \text{ km} \end{aligned}$$

(b) The distance between Kellaton and Harestone can be measured as 4.3 cm.

The actual distance can then be calculated:

$$\begin{aligned} 4.3 \times 50\,000 &= 215\,000 \text{ cm} \\ &= 2150 \text{ m} \\ &= 2.15 \text{ km} \end{aligned}$$

Worked Example 2

The distance between two places is 12 km. A map scale is 1 : 25 000. Find the distance between the two places on the map, in centimetres.

Solution

This map will use 1 cm for every 25 000 cm in reality.

First convert 12 km to centimetres.

$$\begin{aligned} 12 \text{ km} &= 12\,000 \text{ m} \\ &= 1\,200\,000 \text{ cm} \end{aligned}$$

To find the distance on the map divide 1 200 000 by 25 000. This gives

$$\frac{1\,200\,000}{25\,000} = 48$$

So on the map the two places will be 48 cm apart.

Worked Example 3

Two places are 4.5 km apart. On a map they are 15 cm apart. What is the scale of the map?

Solution

First convert 4.5 km to cm, so that both distances are given in the same units.

$$\begin{aligned} 4.5 \text{ km} &= 4500 \text{ m} \\ &= 450\,000 \text{ cm} \end{aligned}$$

The scale can then be expressed as the ratio

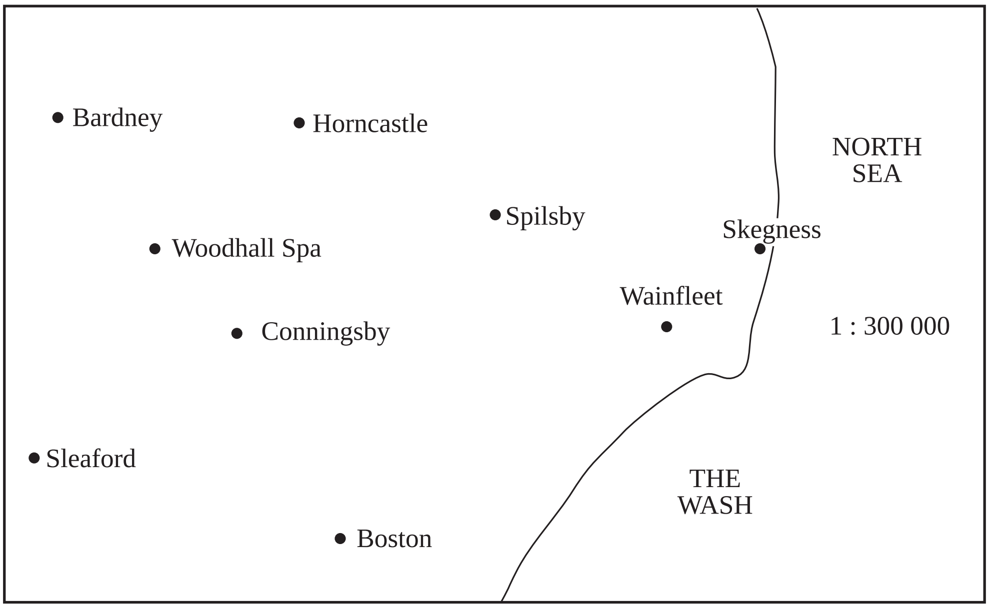
15 : 450 000

because 15 cm on the map represents 450 000 cm in reality. This ratio can be simplified by dividing both numbers by 15 to give

1 : 30 000

Exercises

1. The map below has a scale of 1 : 300 000.



Use the map to find the distances between:

(a) Woodhall Spa and Boston,

(b) Sleaford and Skegness,

(c) Horncastle and Boston,

(d) Spilsby and Conningsby.

2. A map has a scale of 1 : 400 000. Copy and complete the table below which gives the distances between various towns.

Towns	*Distance on map*	*Actual distance*
London and Sheffield	58.5 cm	km
Shrewsbury and Birmingham	15.8 cm	km
Leeds and Hull	21.0 cm	km
Manchester and Newcastle	42.8 cm	km

3. On a map of the Shetland Islands with a scale of 1 : 600 000 the distance between the towns of Scalloway and Lerwick is 0.4 cm. What is the actual distance between the two towns?

4. A ship is to sail from Holyhead in Wales to Dublin in Ireland. On a map with a scale of 1 : 3 000 000, the distance between the two ports is 3.5 cm. Find the actual distance between the ports in kilometres.

5. The direct distance between Dover and Calais is 42 km. What would be the distance between these two ports on maps with scales:

   (a) 1 : 3 000 000, (b) 1 : 1 000 000, (c) 1 : 50 000?

6. On a map with a scale of 1 : 3 000 000 the distance between Bristol and Bath is 0.6 cm. Find the actual distance between the two places and the distance between them on a map with a scale of 1 : 60 000.

7. On a map with a scale of 1 : 300 000 the distance between Burnley and Blackburn is 5.5 cm.

   (a) Find the distance between the two towns in km.

   (b) How far apart would the two towns be on a map with a scale of 1 : 50 000?

8. Cambridge and Huntingdon are 24 km apart. Find the scale of a map that represents this distance by:

   (a) 6 cm, (b) 60 cm, (c) 40 cm, (d) 5 cm.

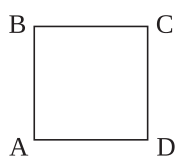
9. A road atlas has maps on each page. The width of each page is 18 cm and this represents a distance of 63 km. Find the scale of the map.

   Some pages of the road atlas are photocopied and their size changed. Find the scale of the map that is produced if the widths of the pages are reduced to:

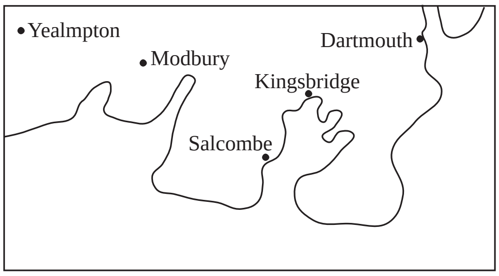
   (a) 9 cm, (b) 12 cm, (c) 16 cm.

10. A grid square on an ordnance survey map has sides of length 2 cm. The map has a scale of 1 : 5000.

    Four points are marked on the ground at the corners of a grid square. Find the actual distance between AC, correct to the nearest metre.



11. (a) Find the scale of the map shown below if the actual distance between Kingsbridge and Salcombe is 5.4 km.



(b) Find the actual distance between Modbury and Dartmouth.

12. On a map with a scale of 1 : 25 000 a plot of land is represented by a rectangle 1.5 cm by 1.2 cm. Find the area of the plot of land.

15.4 Proportional Division

Sometimes a quantity has to be divided in a certain ratio. For example, two waiters may divide their tips in the ratio 2 : 3 because one has worked longer than the other. If they had £5 of tips, one would get £2 and the other £3. If they had £20 of tips, one would get £8 and the other £12.

Worked Example 1

Medrine and Nikki earn £285 by making curtains. Because Medrine did more of the work they decide to divide the £285 in the ratio 3 : 2. How much do they earn each?

Solution

This problem is solved by dividing the £285 into 5 parts and giving 3 parts to Medrine and 2 parts to Nikki. It is divided into 5 parts because the ratio is 3 : 2.

$$\frac{285}{5} = £57$$

$$\begin{aligned} \text{Medrine's share} &= 3 \times 57 \\ &= £171 \end{aligned}$$

$$\begin{aligned} \text{Nikki's share} &= 2 \times 57 \\ &= £114 \end{aligned}$$

Worked Example 2

Pineapple, orange and apple juices are mixed in the ratio 2 : 3 : 5 to make a new drink. Find the volume of each type of juice contained in 250 cm^3 of the new drink.

Solution

Adding the terms of the ratio gives

$$2 + 3 + 5 = 10$$

So the volume of the drink must be divided into 10 parts.

$$\frac{250}{10} = 25 \text{ cm}^3$$

Now the volume of each type of juice can be calculated.

$$\begin{aligned} \text{Volume of pineapple juice} &= 2 \times 25 \\ &= 50 \text{ cm}^3 \\ \text{Volume of orange juice} &= 3 \times 25 \\ &= 75 \text{ cm}^3 \\ \text{Volume of apple juice} &= 5 \times 25 \\ &= 125 \text{ cm}^3 \end{aligned}$$

Exercises

1. The ratio of the volume of oxygen to nitrogen in the air is 1 : 4. Find the volume of oxygen and the volume of nitrogen in 10 litres of air.

2. The ratio of boys to girls in a school play is 2 : 5. Find the numbers of boys and girls in the play if there were 63 children in the play.

3. Ben and Emma collect 140 conkers. They share them out in the ratio 4 : 3. How many conkers do they get each?

4. Claire and Laura work as waitresses. Each week, Claire works on 5 evenings and Laura on 4 evenings. They share any tips in the ration 5 : 4 at the end of each week. How much do they get each if the total of tips for the week is:

   (a) £12.69,

   (b) £33.57,

   (c) £24? (Give the answer to the nearest penny.)

5. In packs of *Fruit and Nut*, raisins and peanuts are mixed. The ratio of the weight of nuts to the weight of raisins is 5 : 3. Find the weight of nuts and the weight of raisins in:

   (a) 800 grams,

   (b) 200 grams,

   (c) 300 grams of the mix.

6. A football team arranges for £1000 to be divided among its players in the ratio of the goals scored. Goals are scored by the three players listed below.

Tim Nicholson	5
Ben Townsend	6
Kris Penk	9

How much cash does each player get?

7. Hannah, Adam, Lucy and Jacob are left £20 000 by a long lost relative. They divide the money in the ratio 4 : 3 : 2 : 1. How much does each of the children get?

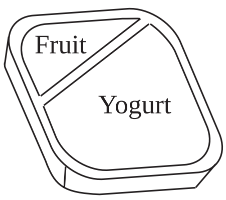
8. Ahmed and Afzal win a jar containing 500 sweets in a competition. They divide the sweets in the ratio 3 : 7. Ahmed shares his in the ratio 3 : 2 with his wife and Afzal shares his is the ratio 3 : 4 with his girlfriend.

   How many sweets do Ahmed and Afzal have each?

9. Apples, bananas and oranges are mixed in the ratio 5 : 6 : 4 respectively by weight, to make a fruit salad. What weight of each type of fruit would be needed to make 6 kg of fruit salad?

10. A music shop sells a total of 240 albums in one day. Some are on cassette and some on CD. The ratio of CDs to cassettes is 1 : 2. For the cassettes the ratio of pop to classical sales is 7 : 1. For CDs the ratio of classical to pop is 1 : 3. Find the total number of classical albums sold.

11. A fruit corner yogurt weighs 175 g altogether.



The ratio of the **weight of fruit** to the **weight of yogurt** is 2 : 5.

Calculate the weight of the fruit.

(SEG)

Just For Fun

A cashier of a bank was given one million one pound coins to count. How long will he take if he can count five coins in one second?

15.5 Direct Proportion

If two quantities increase or decrease at the same rate, they are said to be *directly proportional*. For example, the table shows the distance travelled by a cyclist and the number of revolutions of the wheels of the bicycle.

Distance travelled (m)	18	36	90	180	360
Number of revolutions	10	20	50	100	200

Note how, if one quantity is doubled, then the other is also doubled. The distance travelled is directly proportional to the number of revolutions. This can be written as

$$d \propto N$$

where d is the distance and N is the number of revolutions. The relationship between the two quantities can be written as $d = kN$ where k is a constant.

In fact, $d = 1.8N$ in this example.

The number 1.8 is called the *constant of proportionality.*

Worked Example 1

The table shows values of the quantities x and y. If y is directly proportional to x, find the relationship between x and y and fill in the missing numbers in the table.

x	3	6	8		
y	15			80	120

Solution

As y is directly proportional to x, the relationship must be of the form $y = kx$. To find the value of k use the first two values, $x = 3$ and $y = 15$. This gives

$$15 = k \times 3,$$

$$\text{so} \qquad k = \frac{15}{3}$$

$$= 5$$

$$\text{and} \qquad y = 5x$$

Now the missing values can be found.

Using $x = 6$ gives

$$y = 5 \times 6$$
$$= 30$$

Using $x = 8$ gives

$$y = 5 \times 8$$
$$= 40$$

Using $y = 80$ gives

$$80 = 5x$$

$$x = \frac{80}{5}$$

$$x = 16$$

Using $y = 120$ gives

$$120 = 5x$$

$$x = \frac{120}{5}$$

$$x = 24$$

So the completed table is

x	3	6	8	16	24
y	15	30	40	80	120

Worked Example 2

The table gives values for the quantities P and Q.

P	5	11	12
Q	35	77	81

Is Q directly proportional to P?

Solution

If Q is directly proportional to P, then $Q = kP$ and $k = \frac{Q}{P}$.

If $\frac{Q}{P}$ is the same for each pair of values then Q will be proportional to P.

Using $P = 5$ and $Q = 35$ gives $\frac{Q}{P} = \frac{35}{5}$
$= 7$

Using $P = 11$ and $Q = 77$ gives $\frac{Q}{P} = \frac{77}{11}$
$= 7$

Using $P = 12$ and $Q = 81$ gives $\frac{Q}{P} = \frac{81}{12}$

$$= 6.75$$

As these values are not all the same, Q is not directly proportional to P.

Worked Example 3

The age of a certain species of tree is directly proportional to the diameter of its trunk. A 4-year-old tree has a trunk diameter of 12 mm.

(a) Find the relationship between the age and the trunk diameter.

(b) Find the age of a tree with a trunk diameter of 276 mm.

(c) Find the trunk diameter of a tree 50 years old.

Solution

(a) Let A be the age of the tree and d the diameter of its trunk.

As the age is directly proportional to the diameter of the trunk we can write

$$A \propto d$$

$$\text{or } A = kd$$

Using $A = 4$ and $d = 12$ gives

$$4 = k \times 12$$

$$k = \frac{4}{12}$$

$$= \frac{1}{3}$$

So the relationship is: $A = \frac{1}{3} \times d$

(b) Using the relationship with $d = 276$ gives

$$A = \frac{1}{3} \times 276$$

$$= 92 \text{ years}$$

(c) Using the relationship with $A = 50$ gives

$$50 = \frac{1}{3} \times d$$

$$d = 3 \times 50$$

$$= 150 \text{ mm}$$

Exercises

1. Using the data given in each table check whether the statement given could be correct.

(a)

x	3	7	11
y	39	91	143

$y \propto x$

(b)

p	8	11	52
q	9.6	13.2	62.4

$q \propto p$

(c)

r	7	8	9
s	11.2	13.6	14.4

$s \propto r$

(d)

x	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
y	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{40}$

$y \propto x$

2. Copy and complete each table using the statement beside it.

(a)

x	8	10	15	22
y	24			

$y \propto x$

(b)

p	5	7	9	11
q		42		

$q \propto p$

(c)

r	2	3	5	7
s			30	

$s \propto r$

(d)

w	4	8	9	
z	28			84

$z \propto w$

(e)

x	5	6	7	
y	17.5			31.5

$y \propto x$

(f)

t	3	4		
h	29.4		58.8	88.2

$h \propto t$

(g)

p				7
q	19.8	26.4	38.5	77

$q \propto p$

(h)

v		12		
p	1.80	4.32	6.84	10.44

$p \propto v$

3. The cost, C, of building the roof of a house is proportional to the area, A, it has to cover. A roof costs £6000 and covers an area of 36 m^2.

   (a) Find the relationship between C and A.

   (b) Find the cost of a roof to cover an area of 27 m^2.

   (c) A roof costs £7500. What area does it cover?

4. A spring stretches when a force is applied to one end. The extension, x, of the spring is proportional to the size of the force, F. When the force is 20 N the extension is 8 cm.

   (a) Find the relationship between x and F.

   (b) What extension would be produced by a force of 25 N?

   (c) What force is needed to produce an extension of 13 cm?

5. A bicycle has wheels with a radius of 30 cm. The distance, d, travelled by the bicycle is proportional to the number of revolutions, n, of the wheels.

   (a) Find an expression of d in terms of n.

   (b) How far would the bicycle travel if the wheels completed 80 revolutions?

   (c) How many times would the wheels go round if the bicycle travelled 2 km?

6. A car travels at a constant speed of 70 mph along a motorway. The distance, d, travelled is proportional to the time, t, the car has been driven.

   (a) Find a relationship between d and t.

   (b) How far does the car travel in:

   (i) 2 hours, (ii) 5 mins, (iii) 1.2 hours?

   (c) How long does it take the car to travel:

   (i) 210 miles, (ii) 14 miles, (iii) 2 miles?

7. The circumference of a circle is proportional to the radius.

   (a) Write down the formula for the circumference of a circle and state the value of the constant of proportionality.

   (b) It is also possible to state that the circumference of a circle is proportional to the diameter. What would be the constant of proportionality in this case?

8. The quantities x, y and z are such that $y \propto x$ and $z \propto y$. If $y = 30$ when $x = 5$ and $z = 28$ when $y = 20$, find the relationship between x and z.

15.6 Inverse Proportion

Two quantities may vary so that when one is doubled the other is halved. This is an example where the quantities are said to be *inversely proportional*. For example, consider how long it takes to complete a 300 km journey when travelling at different speeds.

Speed (km/hr)	100	60	50	40	30
Time (hours)	3	5	6	7.5	10

Note that if the speed is halved from 100 km/h to 50 km/h, the time is doubled from 3 hours to 6 hours. The time taken is inversely proportional to the speed.
Also note that the speed × time is always equal to 300. If we use v for speed and t for time, we can write:

$$vt = 300$$

or

$$t = \frac{300}{v}$$

If y is inversely proportional to x, we write

$$y \propto \frac{1}{x}$$

or

$$y = \frac{k}{x}$$

where k is a constant.

Worked Example 1

Determine whether the relationship given with each table is true or false.

(a)

x	8	12	24
y	6	4	2

$$y \propto \frac{1}{x}$$

(b)

p	10	7	5
q	3	6	8

$$q \propto \frac{1}{p}$$

Solution

(a) If $y \propto \frac{1}{x}$, then $x \times y$ will always give the same value.

$$8 \times 6 = 48$$

$$12 \times 4 = 48$$

$$24 \times 2 = 48$$

As the value 48 is always obtained, $y \propto \frac{1}{x}$ and in fact, $xy = 48$ or $y = \frac{48}{x}$.

(b) If $q \propto \dfrac{1}{p}$, then $p \times q$ will always give the same value.

$$10 \times 3 = 30$$

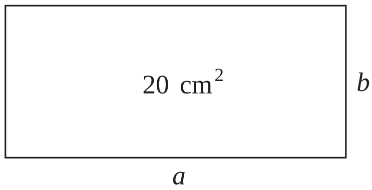
$$7 \times 6 = 42$$

$$5 \times 8 = 40$$

As these values are not all the same, q is not inversely proportional to p.

Worked Example 2

The rectangle in the diagram has an area of 20 cm^2.
Show that a is inversely proportional to b.



Solution

As the area is 20 cm^2,

$$a \times b = 20 \text{ cm}^2$$

or $$a = \frac{20}{b} \text{ cm}$$

So $a \propto \dfrac{1}{b}$, that is, a is inversely proportional to b.

Worked Example 3

Copy and complete the table below if $y \propto \dfrac{1}{x}$.

x	2	5	6		
y		12		4	3

Solution

As $y \propto \dfrac{1}{x}$, the relationship between x and y will be of the form

$$y = \frac{k}{x}$$

Using the values $y = 12$ and $x = 5$ allows the value of k to be found.

$$12 = \frac{k}{5}$$

$$k = 12 \times 5$$

$$= 60$$

So $$y = \frac{60}{x}$$

If $x = 2$, the relationship gives,

$$y = \frac{60}{2}$$
$$= 30$$

If $x = 6$, the relationship gives,

$$y = \frac{60}{6}$$
$$= 10$$

If $y = 4$, the relationship gives,

$$4 = \frac{60}{x}$$
$$4x = 60$$
$$x = \frac{60}{4}$$
$$= 15$$

If $y = 3$, the relationship gives,

$$3 = \frac{60}{x}$$
$$3x = 60$$
$$x = \frac{60}{3}$$
$$= 20$$

The table can then be completed as shown below.

x	2	5	6	15	20
y	30	12	10	4	3

Exercises

1. For each table of values below, determine whether the relationship given is true or false.

(a)

x	3	4	5
y	8	6	4

$$y \propto \frac{1}{x}$$

(b)

q	2	4	8
p	14	7	3.5

$$p \propto \frac{1}{q}$$

(c)

r	2	3	7
s	42	28	12

$$s \propto \frac{1}{r}$$

(d)

x	$\frac{1}{4}$	$\frac{1}{2}$	1
y	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{20}$

$$y \propto \frac{1}{x}$$

2. Use the relationship given beside each table to complete a copy of the table.

(a)

x	10	20	25	50
y	10			

$y \propto \frac{1}{x}$

(b)

p	32	8	4	$\frac{1}{2}$
q	2			

$q \propto \frac{1}{p}$

(c)

r	1	3	5	10
s		15		

$s \propto \frac{1}{r}$

(d)

a	0.9	2.4	4.8	
b		0.6		0.2

$b \propto \frac{1}{a}$

(e)

g	1	2	9	
h		8.1		0.2

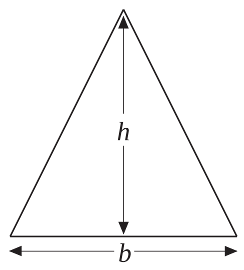
$h \propto \frac{1}{g}$

(f)

u	2			
v	6.8	3.4	1.7	0.8

$v \propto \frac{1}{u}$

3.



The triangle shown has an area of 8 cm^2.

Show that $b \propto \frac{1}{h}$, and state the constant of proportionality.

4. In an electric circuit the current, I, is inversely proportional to the resistance, R, of the circuit.

(a) If, when $R = 1000$, $I = 0.5$, find the relationship between I and R.

(b) If the resistance increases to 1500, what is the current?

(c) What happens to I if R is doubled?

5. The volume, V, of a gas at a constant temperature is inversely proportional to the pressure, P. When the pressure is 100 N m^{-2}, the volume is 8 litres.

   (a) What happens to the volume if the pressure is doubled?

   (b) What happens to the volume if the pressure in increased by a factor of 4?

   (c) Find the relationship between volume and pressure.

   (d) Find the pressure that gives a volume of 10 litres.

6. Two quantities, x and y, are such that y is inversely proportional to x. Increasing x from its initial value to 7 causes the value of y to drop from 8 to 4.

   (a) Find the initial value of x.

   (b) Find the formula that expresses y in terms of x.

   (c) If y had dropped from 8 to 2, what would have happened to x?

7. A second-hand car dealer suspects that the value of a car is inversely proportional to its age. A car was sold for £8000 when it was 3 years old and again for £6000 when it was 4 years old.

   (a) Is it possible that the value is inversely proportional to the age?

   (b) If the value is inversely proportional to the age, find

      (i) the value of the car after 10 years,

      (ii) when the value drops below £1000.

8. The wavelength of sound waves is inversely proportional to their frequency. When the wavelength is 0.25 m the frequency is 1360 Hz.

   (a) Find the frequency when the wavelength is 0.8 m.

   (b) Find the wavelength when the frequency is 850 Hz.

15.7 Functional and Graphical Representations

This section considers further proportionality, involving higher powers such as x^2, x^3, etc. The graphical representations of proportional relationships are also considered.

Worked Example 1

A company produces chocolate sweets shaped and wrapped to look like footballs. The cost of making each sweet is proportional to its radius, squared. The cost of a sweet of radius 3 cm is 4p. Find the cost of a sweet of radius 1.2 cm and the radius of a sweet that costs 8p to make.

Solution

Let C = cost and r = radius.

Then as the cost is proportional to the radius squared,

$$C \propto r^2$$

or

$$C = kr^2$$

Using $C = 4$ when $r = 3$ gives

$$4 = k \times 3^2$$

$$4 = k \times 9$$

$$k = \frac{4}{9}$$

The formula for cost is then

$$C = \frac{4}{9} r^2$$

For a sweet with radius 1.2 cm,

$$C = \frac{4}{9} \times 1.2^2$$

$$= 0.64\text{p}$$

For a sweet which costs 8p to make,

$$8 = \frac{4}{9} \times r^2$$

$$r^2 = \frac{9 \times 8}{4}$$

$$r^2 = 18$$

$$r = \sqrt{18}$$

$$= 4.24 \text{ cm} \quad \text{(to 2 decimal places)}$$

Worked Example 2

The force exerted by a magnet on a metal sphere is inversely proportional to the distance between them. When the distance between them is 20 cm, the force exerted by the magnet is 0.5 N.

(a) Express the force, F, exerted by the magnet in terms of x, the distance between the magnet and the sphere.

(b) Find the force when the magnet is 10 cm from the sphere.

(c) How far is the magnet from the sphere when the force is 20 N?

Solution

(a) The force, F, is inversely proportional to the distance, x, squared.

$$F \propto \frac{1}{x^2}$$

or $$F = \frac{k}{x^2}$$

Using $F = \frac{1}{2}$ when $x = 20$ gives

$$\frac{1}{2} = \frac{k}{20^2}$$

$$k = \frac{20^2}{2}$$

$$= 200$$

So $$F = \frac{200}{x^2}$$

(b) When $x = 10$,

$$F = \frac{200}{10^2}$$

$$= \frac{200}{100}$$

$$= 2 \text{ N}$$

(c) If $F = 20$,

$$20 = \frac{200}{x^2}$$

$$x^2 = \frac{200}{20}$$

$$x^2 = 10$$

$$x = \sqrt{10}$$

$$= 3.16 \text{ cm} \quad \text{(to 2 decimal places)}$$

Worked Example 3

Sketch graphs to show how y varies with x for each of the following relationships.

(a) y is proportional to x,

(b) y is proportional to x^2,

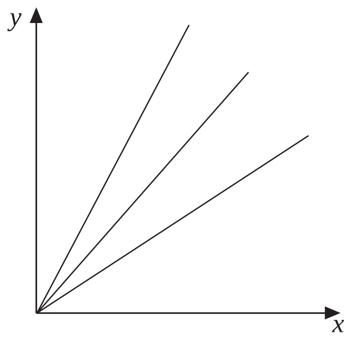
(c) y is inversely proportional to x.

Solution

(a) If y is proportional to x,

$$y = kx$$

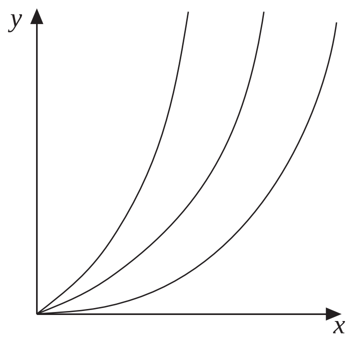
This is the equation of a straight line that passes through (0, 0), as when $x = 0$, $y = 0$. The greater the value of k the steeper the line will be. The graph shows $y = kx$ for three possible values of k.



(b) If y is proportional to x^2, then

$$y = kx^2$$

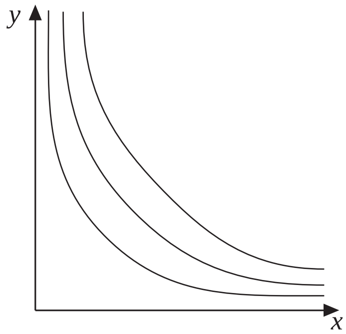
This gives graphs as shown below, with the steeper graphs having greater values of k.



Note that each curve passes through the point (0, 0)

(c) If y is inversely proportional to x, then

$$y = \frac{k}{x}$$



This graph shows 3 curves. The smaller the value of k, the closer the curve is to the axes.

Exercises

1. Write down a relationship between each pair of variables, using the data given to find the constant of proportionality.

   (a) T is proportional to x, and when $x = 3$, $T = 120$.

   (b) P is proportional to the square of v, and when $v = 2$, $P = 160$.

   (c) R is inversely proportional to the cube of x, and when $x = 3$, $R = 8$.

   (d) Y is inversely proportional to x, and when $x = 3$, $Y = 24$.

   (e) V is inversely proportional to the square of x, and when $x = 5$, $V = 100$.

2. Express each of the following statements in words.

   (a) $y \propto \frac{1}{x^5}$ (b) $y \propto x^2$ (c) $y \propto \frac{1}{x^2}$

   (d) $y \propto x$ (e) $y \propto x^3$ (f) $y \propto \frac{1}{x}$

3. Complete each table below, given the relationship between x and y.

   (a) $y \propto x^2$

x	1	2	5	8
y		16		

   (b) $y \propto x^3$

x	1	1.1	1.2	1.3
y	2			

   (c) $y \propto \frac{1}{x^2}$

x		2	5	
y	5	1.25		0.05

   (d) $y \propto \frac{1}{x^3}$

x	1	3		
y		1	$\frac{1}{27}$	$\frac{1}{64}$

4. If y is proportional to the cube of x, and $y = 12$ when $x = 2$, find:

   (a) y when $x = 8$,

   (b) x when $y = 96$.

5. The volume of a cube is proportional to the cube of the length of a side. What is the constant of proportionality?

6. The volume, V, of a balloon is proportional to the cube of its radius, r. When the radius of 10 cm the volume is 4200 cm^3.

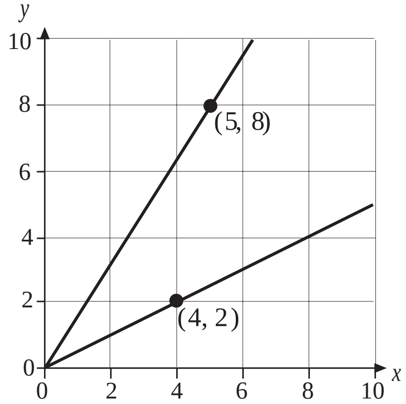
   (a) Find a relationship between V and r.

   (b) Find the volume of a balloon of radius 5 cm.

   (c) Find the radius of a balloon that has a volume of 8400 cm^3.

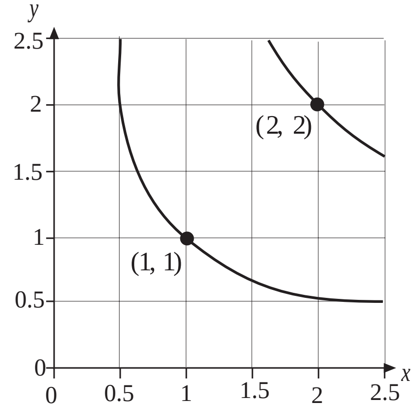
7. The graphs below show possible relationships between x and y. Find the value of k that corresponds to each line or curve, given the relationship below the graph.

(a)



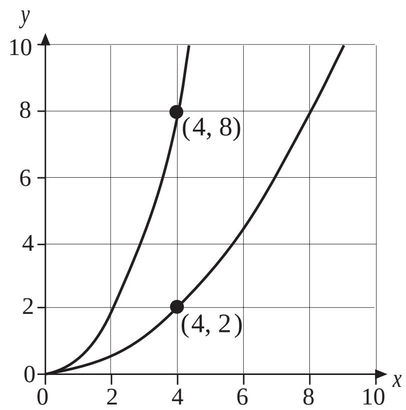
$$y = kx$$

(b)



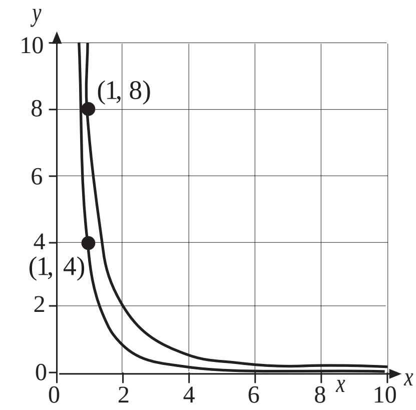
$$y = \frac{k}{x}$$

(c)



$$y = kx^2$$

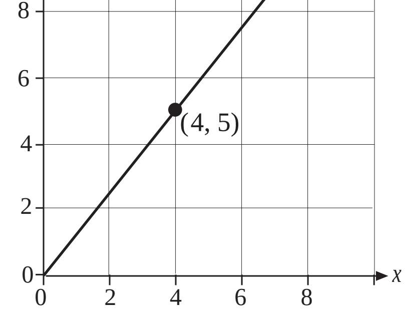
(d)



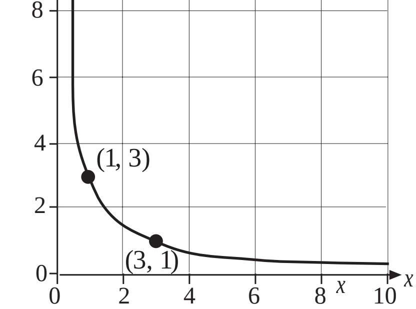
$$y = \frac{k}{x^2}$$

8. For each graph below $y \propto x^n$ where n is 1, 2, –1 or –2. Use the information given to find n and the relationship between y and x.

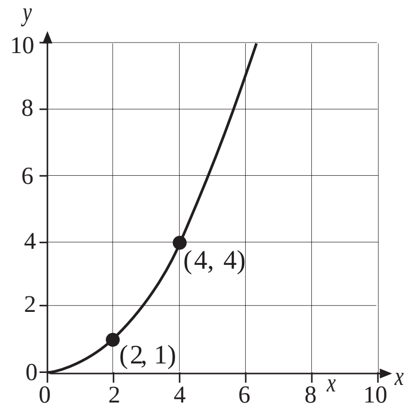
(a)



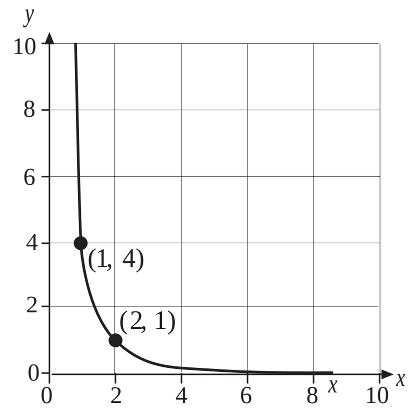
(b)



(c)



(d)



9. If y is inversely proportional to x and z is proportional to y^2, what is the relationship between z and x? Find the constant of proportionality if $y = 2$ when $x = 4$ and $z = 9$ when $y = 2$.

10. The intensity of the light from a projector falling on a screen is inversely proportional to the square of the distance between the projector and the screen.

    (a) How does doubling the distance between the screen and the projector affect the light intensity?

    (b) How should the distance between the screen and the projector be adjusted to double the light intensity?

15.8 Further Functional Representations

Statements involving proportionality can be more complex than those met so far; for example,

'y is inversely proportional to the square root of x.'

Worked Example 1

When a mass suspended from a spring oscillates, the frequency of the oscillations is inversely proportional to the square root of the mass. A particular system has a frequency of 2 Hz with a mass of 0.25 kg. Find the mass required to produce a frequency of 8 Hz.

Solution

Let f = frequency and m = mass.

We have

$$f \propto \frac{1}{\sqrt{m}}$$

or

$$f = \frac{k}{\sqrt{m}}$$

Using $f = 2$ when $m = 0.25$ gives

$$2 = \frac{k}{\sqrt{0.25}}$$

$$2 = \frac{k}{0.5}$$

$$k = 2 \times 0.5$$

$$= 1$$

So $$f = \frac{1}{\sqrt{m}}$$

If $f = 8$

then $$8 = \frac{1}{\sqrt{m}}$$

so $$\sqrt{m} = \frac{1}{8}$$

and $$m = \frac{1}{8^2}$$

$$= 0.015625 \text{ kg}$$

Worked Example 2

The period of a simple pendulum is proportional to the square root of the length of the pendulum. How should the length of the pendulum be changed to double the period?

Solution

Let T = period and l = length of the original pendulum. As $T \propto \sqrt{l}$ we have $T = k\sqrt{l}$.

Let x be the length of the pendulum with double the period.

Then

$$2T = k\sqrt{x}$$

but

$$T = k\sqrt{l}$$

so the equation becomes

$$2k\sqrt{l} = k\sqrt{x}$$

or

$$2\sqrt{l} = \sqrt{x}$$

Squaring both sides gives

$$4l = x$$

So the pendulum must be 4 times longer than the original.

Exercises

1. Write down an expression for each relationship described below, defining suitable variables.

   (a) The fuel consumption of a car is inversely proportional to the square root of its speed.

   (b) The rate at which water runs out of a bath is proportional to the depth of water in the bath.

   (c) The air resistance on a car is proportional to the square root of its speed, cubed.

   (d) The period of the oscillations of a mass suspended from a spring is inversely proportional to the square root of the stiffness of the spring.

2. Sketch graphs to show the relationships between x and y. Sketch both relationships given in each part of the question on the same set of axes so that they can be compared.

   (a) $y \propto x^3$ and $y \propto \dfrac{1}{x^3}$ (b) $y \propto \sqrt{x}$ and $y \propto \sqrt[3]{x}$

   (c) $y \propto \dfrac{1}{\sqrt{x}}$ and $y \propto \dfrac{1}{x}$ (d) $y \propto \sqrt{x}$ and $y \propto x^2$

3. Copy and complete each table below, using the relationships given.

   (a) $y \propto \sqrt{x}$

x	4	9	100	
y	1			6

   (b) $y \propto \dfrac{1}{\sqrt{x}}$

x	4	9		
y		1	0.6	0.3

   (c) $y \propto \left(\dfrac{1}{\sqrt{x}}\right)^3$

x	1	4		
y	5		$\dfrac{5}{27}$	$\dfrac{5}{343}$

   (d) $y \propto \sqrt{x^3}$

x	9	25		
y	54		432	1458

4. The period of an oscillating mass–spring system is proportional to the square root of the mass. When the mass is 200 grams the period is 1.2 seconds.

   (a) Find the period for a 300 gram mass.

   (b) What mass is needed to give a period of 2 seconds?

   (c) What happens to the period if the mass is doubled?

   (d) What must happen to the mass to double the period?

5. The length of the side of a cube is proportional to the volume to the power *n*.

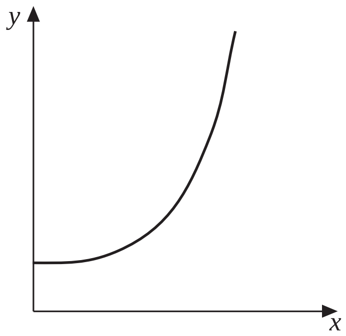
   (a) Find the value of *n*.

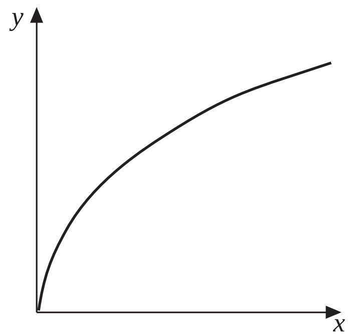
   (b) If the volume of the cube is to be halved, what must happen to the lengths of the sides?

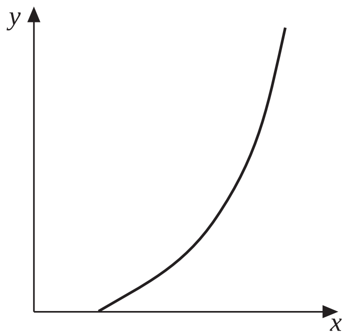
   (c) What happens to the lengths of the sides if the volume is doubled?

6. A zoologist assumes that the mass of an elephant is proportional to the cube of its height. A young elephant has a height of 1.5 m and a mass of 400 kg. Find the mass of the elephant when it has grown to a height of 2.4 m

7. For each of the following graphs, state whether or not $y \propto x^n$. Also state what can be deduced about the value of *n*, i.e. positive or negative, greater than 1, etc.

(a)


(b)


(c)


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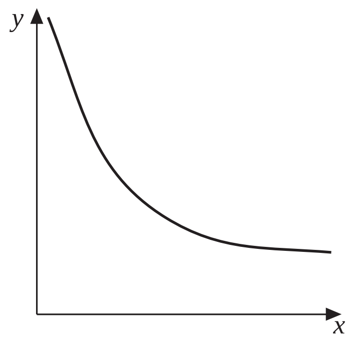

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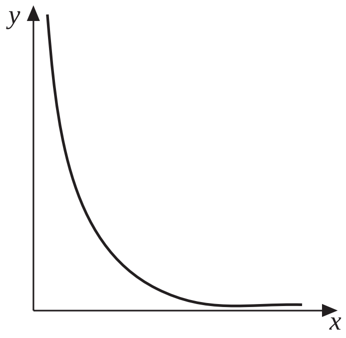
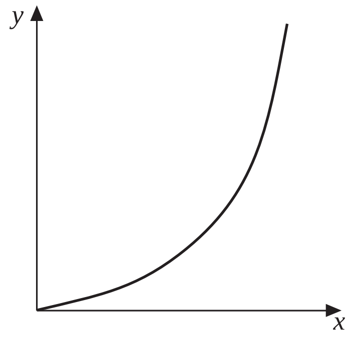


(f)


8. (a) Write down a formula for the volume of a cylinder.

   (b) If the radius is fixed, write down a statement to describe how the volume varies with the height of the cylinder, in the form $V \propto \ldots$.

   (c) If the height is fixed, write down a statement to describe how the volume varies with the radius of the cylinder, in the form $V \propto \ldots$.

9. The period, T, of a mass–spring system is inversely proportional to the square root of the stiffness, k, of the spring and inversely proportional to the square root of the mass, m. Write down a single statement of proportionality for the period in the form $T \propto ...$.

10. If $y \propto x^a$, $z \propto y^b$ and $z \propto x^n$, find n if:

(a) $a = 2$ and $b = 3$

(b) $a = \frac{1}{2}$ and $b = -2$

(c) $a = 3$ and $b = -\frac{1}{2}$

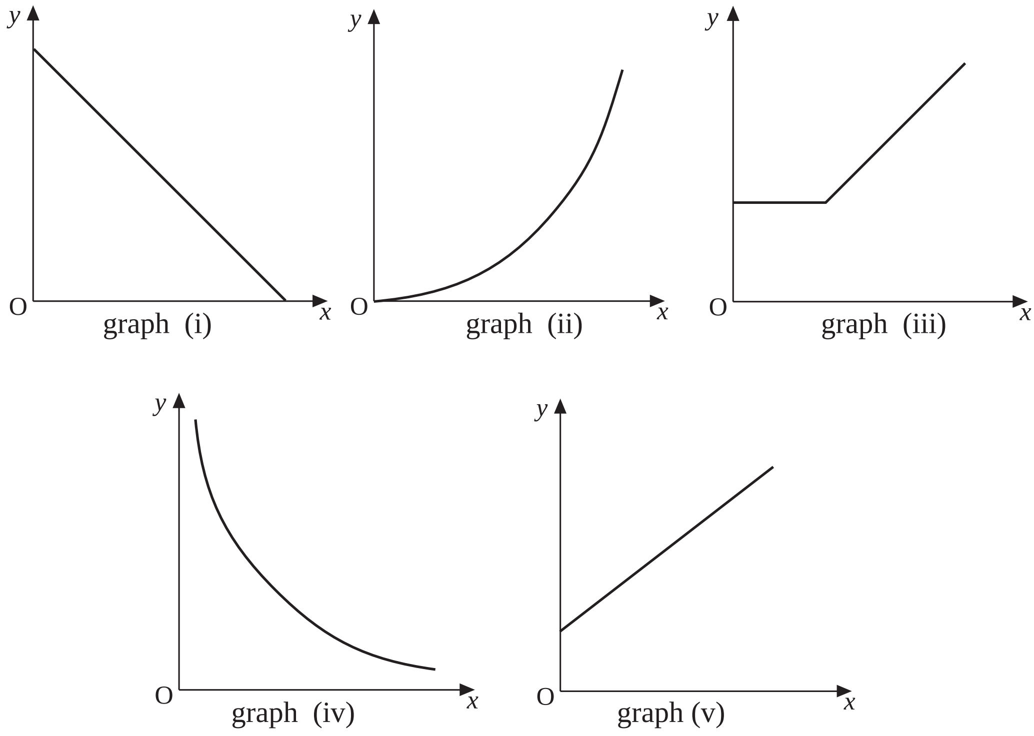
Describe each statement in words.

11. The time of swing, T seconds, of a pendulum is proportional to the square root of the length, L centimetres, of the pendulum.

A pendulum of length 64 centimetres has a time of swing 1.6 seconds.

Find the formula for T in terms of L. *(MEG)*

12.



Select from the five graphs one which illustrates each of the following statements.

(a) The time (y) taken for a journey is inversely proportional to the average speed (x).

(b) The surface area (y) of a sphere is proportional to the square of the radius (x).

(c) The cost (y) of an electricity bill consists of a fixed charge plus an amount proportional to the number of units of electricity used (x).

(MEG)